DIPOLE-DIPOLE FORCES AND TORQUES

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.29.

We'll return to the problem of the interaction of two dipoles that we considered a while back. We have two perfect dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ separated by a distance r. Their alignment is such that $\mathbf{p}_1$ is perpendicular to the line separating them (pointing upwards) and $\mathbf{p}_2$ is parallel to the line separating them (pointing away from $\mathbf{p}_1$). The problem is to find the force that each dipole exerts on the other. We've seen that the force felt by a dipole in an electric field is

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \tag{1}$$

The field in each case is produced by the other dipole, and the formula for such a field is

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right] \tag{2}$$

It's important to be clear on the coordinate definitions in this formula. This is a spherical coordinate system centred on the dipole producing the field, with the vertical, or z, axis parallel to the dipole moment $\mathbf{p}$. The angle θ is the angle measured from the z axis.

First, let's consider the force on $\mathbf{p}_2$ due to $\mathbf{p}_1$. In this case, we centre the coordinate system on $\mathbf{p}_1$. In this system, we can place $\mathbf{p}_2$ so that it points along the $+x$ axis. The force on $\mathbf{p}_2$ is therefore

$$\mathbf{F}_2 = (\mathbf{p}_2\cdot\nabla)\mathbf{E}_1 \tag{3}$$

where $\mathbf{E}_1$ is the field produced by $\mathbf{p}_1$.

It turns out that it's very difficult to work on this problem if we stay in spherical coordinates, so we'll need to switch to rectangular coordinates. In our coordinate system, $\mathbf{p}_2 = p_2\hat{\mathbf{x}}$, so $\mathbf{p}_2\cdot\nabla = p_2\frac{\partial}{\partial x}$. This means that we can hold y and z constant in $\mathbf{E}_1$. Holding z constant means that θ is constant at $\theta = \pi/2$ (since we're in the xy plane), and holding y constant means that $\phi = 0$ (since we're restricting our attention to the x axis). In the xy plane, $\hat{\theta} = -\hat{\mathbf{z}}$. Finally, $r = x$. Therefore, the field we're interested in can be written as

$$\mathbf{E}_1 = -\frac{p_1}{4\pi\epsilon_0 x^3}\hat{\mathbf{z}} \tag{4}$$

From this, we get

$$\mathbf{F}_2 = \frac{3p_1p_2}{4\pi\epsilon_0 x^4}\hat{\mathbf{z}} \tag{5}$$

$$= \frac{3p_1p_2}{4\pi\epsilon_0 r^4}\hat{\mathbf{z}} \tag{6}$$

That is, the force on $\mathbf{p}_2$ is upwards.

Now for the force on $\mathbf{p}_1$. This time, we centre the coordinates on $\mathbf{p}_2$ with the z axis parallel to $\mathbf{p}_2$. In this system, $\mathbf{p}_1$ is at position $-r\hat{\mathbf{z}}$, and it points in the $-x$ direction, so $\mathbf{p}_1 = -p_1\hat{\mathbf{x}}$. Unfortunately, at this stage things don't simplify as much as in the previous case. We get $\mathbf{p}_1 \cdot \nabla = -p_1\frac{\partial}{\partial x}$, which means we want to hold y and z constant, as before, but because $\mathbf{p}_1$ is at $-r\hat{\mathbf{z}}$, varying x will cause changes in both r and θ (although at least we can keep $\phi = 0$ constant). There may be a simpler way of doing this, but one way that does seem to work is to express the formula for $\mathbf{E}$ above entirely in rectangular coordinates.

The standard formulas for converting the unit vectors are:

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \tag{7}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \tag{8}$$

If we set $\phi = 0$ and substitute these formulas into the formula for $\mathbf{E}_2$, we get

$$\mathbf{E}_2 = \frac{p_2}{4\pi\epsilon_0 r^3}\left[2\cos\theta\left(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}\right) + \sin\theta\left(\cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{z}}\right)\right] \tag{9}$$

$$= \frac{p_2}{4\pi\epsilon_0 r^3}\left[3\sin\theta\cos\theta\hat{\mathbf{x}} + \left(3\cos^2\theta - 1\right)\hat{\mathbf{z}}\right] \tag{10}$$

In the case $\phi = 0$, we have

$$\sin\theta = \frac{x}{r} \tag{11}$$

$$\cos\theta = \frac{z}{r} \tag{12}$$

so:

$$\mathbf{E}_2 = \frac{p_2}{4\pi\epsilon_0 r^5}\left[3xz\hat{\mathbf{x}} + \left(3z^2 - r^2\right)\hat{\mathbf{z}}\right] \tag{13}$$

Using the product rule and $r=\sqrt{x^2+y^2+z^2}$, we get

$$\mathbf{F}_1=-p_1\frac{\partial}{\partial x}\mathbf{E}_2=\frac{5p_1p_2x}{4\pi\epsilon_0r^7}\left[3xz\hat{\mathbf{x}}+\left(3z^2-r^2\right)\hat{\mathbf{z}}\right]-\frac{p_1p_2}{4\pi\epsilon_0r^5}\left[3z\hat{\mathbf{x}}-2x\hat{\mathbf{z}}\right] \tag{14}$$

$$=\frac{3p_1p_2}{4\pi\epsilon_0r^7}\left[\left(5x^2-r^2\right)z\hat{\mathbf{x}}+\left(5z^2-r^2\right)x\hat{\mathbf{z}}\right] \tag{15}$$

We need to evaluate this at $x=y=0;\ z=-r$, so we get

$$\mathbf{F}_1=\frac{3p_1p_2}{4\pi\epsilon_0r^4}\hat{\mathbf{x}} \tag{16}$$

With the orientation of our coordinate system, the $+x$ direction is downwards, so $\mathbf{F}_1=-\mathbf{F}_2$, which is consistent with Newton's third law of equal and opposite forces.

We can now work out the torques on the two dipoles relative to each other. In our original analysis of this problem we found that the torques exerted by one dipole on the other were not symmetric, with the torque on $\mathbf{p}_1$ being twice that on $\mathbf{p}_2$. However, each torque was calculated relative to the centre of the respective dipole, so we weren't comparing like with like. What we really need to do is compare the torques on the dipoles *with respect to the same point*. For example, since we worked out the torque on $\mathbf{p}_1$ relative to its centre as

$$\mathbf{N}_1=\frac{2}{4\pi\epsilon_0r^3}\mathbf{p}_1\times\mathbf{p}_2 \tag{17}$$

we need to work out the torque on $\mathbf{p}_2$, also relative to the centre of $\mathbf{p}_1$.

A quick reminder of how to transform torques from one axis to another. Suppose that the dipole is slightly non-ideal, in that there is a finite separation between the two charges. Call the vector from the centre of the dipole (the midpoint between the charges) to one of the charges $\mathbf{r}_a$ and the vector from the centre to the other charge $\mathbf{r}_b$. If charge a has a force $\mathbf{F}_a$ acting on it, and charge b has force $\mathbf{F}_b$, then the torque on the dipole is

$$\mathbf{N}_{centre}=\mathbf{r}_a\times\mathbf{F}_a+\mathbf{r}_b\times\mathbf{F}_b \tag{18}$$

Now suppose we want the torque due to these same two forces, but about a different point O. Call the vector from O to the midpoint of the dipole $\mathbf{s}$. Then the vectors from O to charges a and b are

$$\mathbf{s}_a = \mathbf{s}+\mathbf{r}_a \tag{19}$$

$$\mathbf{s}_b = \mathbf{s}+\mathbf{r}_b \tag{20}$$

The torque relative to O is therefore

$$\mathbf{N}_O = \mathbf{s}_a \times \mathbf{F}_a + \mathbf{s}_b \times \mathbf{F}_b \tag{21}$$
$$= \mathbf{s} \times (\mathbf{F}_a + \mathbf{F}_b) + \mathbf{r}_a \times \mathbf{F}_a + \mathbf{r}_b \times \mathbf{F}_b \tag{22}$$
$$= \mathbf{s} \times (\mathbf{F}_a + \mathbf{F}_b) + \mathbf{N}_{centre} \tag{23}$$

That is, the torque is the sum of the original torque and a new torque consisting of the vector product of $\mathbf{s}$ and the total force acting on the dipole.

In terms of our ideal dipoles, the torque on $\mathbf{p}_2$ relative to $\mathbf{p}_1$ is then

$$\mathbf{N}_{21} = \mathbf{r} \times \mathbf{F}_2 + \mathbf{N}_2 \tag{24}$$

In our earlier analysis, we found that

$$\mathbf{N}_2 = \frac{1}{4\pi\epsilon_0 r^3}\mathbf{p}_1 \times \mathbf{p}_2 \tag{25}$$

Using the coordinate system centred on $\mathbf{p}_1$,

$$\mathbf{N}_2 = \frac{p_1 p_2}{4\pi\epsilon_0 r^3}\hat{\mathbf{y}} \tag{26}$$
$$\mathbf{r} \times \mathbf{F}_2 = -\frac{3p_1 p_2}{4\pi\epsilon_0 r^3}\hat{\mathbf{y}} \tag{27}$$
$$\mathbf{N}_{21} = -\frac{2p_1 p_2}{4\pi\epsilon_0 r^3}\hat{\mathbf{y}} \tag{28}$$

We can see that $\mathbf{N}_{21} = -\mathbf{N}_1$ so that when we measure the torques acting on the two dipoles with respect to the same point, they are in fact equal and opposite, as we would expect.

PINGBACKS

Pingback: Dipole-dipole interactions