POINT CHARGE IN DIELECTRIC SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.32.

Yet another example of the bound charge due to polarization in a dielectric. Suppose we have a point charge at the centre of a sphere of linear dielectric (susceptibility χ_e) of radius R. We can begin the solution by calculating the displacement, since the system has spherical symmetry. We know that

$$\int_V \nabla\cdot\mathbf{D}d^3\mathbf{r} = Q_f \tag{0.1}$$

$$= \int_A \mathbf{D}\cdot d\mathbf{a} \tag{0.2}$$

where Q_f is the free charge enclosed by the surface of integration. Here we can take the surface to be a sphere of radius r centred at the point charge, which is the only free charge in the problem. Therefore

$$\int_A \mathbf{D}\cdot d\mathbf{a} = 4\pi r^2 D(r) \tag{0.3}$$

$$= q \tag{0.4}$$

$$\mathbf{D}(\mathbf{r}) = \frac{q}{4\pi r^2}\hat{\mathbf{r}} \tag{0.5}$$

For a linear dielectric we have

$$\mathbf{D} = \varepsilon_0(1+\chi_e)\mathbf{E} \tag{0.6}$$

so we can get the field

$$\mathbf{E} = \begin{cases} \frac{q}{4\pi\varepsilon_0(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2} & 0<r<R \\ \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2} & r>R \end{cases} \tag{0.7}$$

The polarization, again because we're dealing with a linear dielectric, is

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \tag{0.8}$$

$$= \frac{\chi_e q}{4\pi(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2} \tag{0.9}$$

The bound charges induced by this polarization are

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \tag{0.10}$$

$$\rho_b \equiv -\nabla\cdot\mathbf{P} \tag{0.11}$$

At the surface of the sphere, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and we get

$$\sigma_b = \frac{\chi_e q}{4\pi(1+\chi_e)R^2} \tag{0.12}$$

$$Q_s = 4\pi R^2\sigma_b \tag{0.13}$$

$$= \frac{\chi_e q}{(1+\chi_e)} \tag{0.14}$$

For the bound charge, we make use of the formula for the three-dimensional delta function:

$$\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta_3(\mathbf{r}) \tag{0.15}$$

The bound charge is therefore

$$\rho_b = -\nabla\cdot\mathbf{P} \tag{0.16}$$

$$= -\frac{\chi_e q}{(1+\chi_e)}\delta_3(\mathbf{r}) \tag{0.17}$$

The total volume bound charge is the integral of this over the sphere, which is just

$$Q_v = -\int_V \frac{\chi_e q}{(1+\chi_e)}\delta_3(\mathbf{r})d^3\mathbf{r} \tag{0.18}$$

$$= -\frac{\chi_e q}{(1+\chi_e)} \tag{0.19}$$

All the volume bound charge is concentrated at the centre, and the total bound charge is $Q_v + Q_s = 0$ as required.