ELECTRIC DISPLACEMENT: BOUNDARY CONDITIONS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.33.

We can work out analogs of the boundary conditions on the electric field in the case of the displacement vector **D**. In general

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{1}$$

where **P** is the polarization density.

We've already seen that

$$\nabla \cdot \mathbf{D} = \rho_f \tag{2}$$

where ρ_f is the free charge density. To apply this to a boundary, suppose we have a boundary between two dielectric materials, with a surface free charge density σ_f at this boundary. Using the same argument as in the case of **E**, we can apply Gauss's law over a small box that straddles the boundary and whose surfaces on each side of the boundary are parallel to the boundary layer itself. We'll find that

$$D_{\perp}^{above} - D_{\perp}^{below} = \sigma_f \tag{3}$$

For the parallel component of **D**, we look at the line integral around a rectangle that straddles the boundary, just as we did with **E**. We know from Stokes's theorem that since $\nabla \times \mathbf{E} = 0$ in all electrostatic problems, the parallel component of **E** is continuous across the boundary. However, in general $\nabla \times \mathbf{P} \neq 0$, so we can't conclude that the parallel component of **D** is continuous, and we get

$$D_{\parallel}^{above} - D_{\parallel}^{below} = P_{\parallel}^{above} - P_{\parallel}^{below} \tag{4}$$

As an example, we can work out how much the electric field bends when we cross a boundary between two linear dielectrics, assuming there is no free charge at the boundary. Without free charge, we have from 3:

$$D_{\perp}^{above} = D_{\perp}^{below} \tag{5}$$
$$\epsilon_0\epsilon_a E_{\perp}^{above} = \epsilon_0\epsilon_b E_{\perp}^{below} \tag{6}$$

where $\epsilon_{a,b}$ is the dielectric constant on either side of the boundary.

For the parallel component, we get from 4 (remember we're dealing with a *linear* dielectric):

$$\epsilon_0\epsilon_a E_{\parallel}^{above} - \epsilon_0\epsilon_b E_{\parallel}^{below} = \epsilon_0\chi_a E_{\parallel}^{above} - \epsilon_0\chi_b E_{\parallel}^{below} \tag{7}$$
$$(1+\chi_a) E_{\parallel}^{above} - (1+\chi_b) E_{\parallel}^{below} = \chi_a E_{\parallel}^{above} - \chi_b E_{\parallel}^{below} \tag{8}$$
$$E_{\parallel}^{above} = E_{\parallel}^{below} \tag{9}$$

Thus the second condition just reproduces the original condition on the continuity of the parallel component of the field, so it doesn't really tell us anything new.

We can express the components of the field in terms of the angle between **E** and the normal to the boundary on either side, which we'll call θ_a and θ_b. On the 'above' side, we have

$$E_{\perp}^{above} = E^{above}\cos\theta_a \tag{10}$$
$$E_{\parallel}^{above} = E^{above}\sin\theta_a \tag{11}$$

with similar conditions on the 'below' side, so we can rewrite the equations above as

$$\epsilon_a E_{\perp}^{above} = \epsilon_b E_{\perp}^{below} \tag{12}$$
$$\epsilon_a E^{above}\cos\theta_a = \epsilon_b E^{below}\cos\theta_b \tag{13}$$
$$E^{above}\sin\theta_a = E^{below}\sin\theta_b \tag{14}$$

Dividing the third equation by the second, we get

$$\frac{\tan\theta_a}{\tan\theta_b} = \frac{\epsilon_a}{\epsilon_b} \tag{15}$$

A larger dielectric constant means a larger tangent and thus a larger angle with the normal to the surface, so the field tends to spread out when entering a dielectric with a higher constant.

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