

DIPOLE IN DIELECTRIC SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.34.

Another example of solving a boundary value problem in a system with linear dielectrics. Suppose we have an ideal dipole \mathbf{p} at the centre of a sphere of dielectric (dielectric constant ϵ_r) and radius R . What is the potential at any point (inside or outside the sphere)?

We can use a similar approach to that of the problem of a dielectric cylinder in an electric field. We first specify the boundary conditions. At the surface of the sphere, the potential must be continuous, so we have

$$(0.1) \quad V_{in}(R) = V_{out}(R)$$

As we saw in the cylinder problem, the condition on the normal derivative of the potential is, since on the outside of the sphere, $\epsilon = \epsilon_0$:

$$(0.2) \quad \epsilon \left. \frac{\partial V_{in}}{\partial r} \right|_{r=R} = \epsilon_0 \left. \frac{\partial V_{out}}{\partial r} \right|_{r=R}$$

Since there is no external field, we must have $V \rightarrow 0$ as $r \rightarrow \infty$. Finally, as $r \rightarrow 0$, the potential must behave like that of an ideal dipole, so, assuming that \mathbf{p} points in the $+z$ direction:

$$(0.3) \quad \lim_{r \rightarrow 0} V_{in}(r) = \frac{p \cos \theta}{4\pi \epsilon r^2}$$

Note that we're using ϵ rather than ϵ_0 in this formula, since the dipole is inside the dielectric and the potential is reduced by a factor of ϵ_r , where $\epsilon = \epsilon_0 \epsilon_r$.

With these conditions, we must solve Laplace's equation in spherical coordinates, so we can quote the general form of the solution:

$$(0.4) \quad V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Inside the sphere, we have

$$(0.5) \quad V_{in} = \frac{p \cos \theta}{4\pi\epsilon r^2} + \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

Outside, we have

$$(0.6) \quad V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

Applying the continuity condition 0.1 and equating coefficients of the P_l , we get

$$(0.7) \quad \frac{p}{4\pi\epsilon R^2} + A_1 R = \frac{B_1}{R^2}$$

$$(0.8) \quad A_l R^l = \frac{B_l}{R^{l+1}} \quad (l \neq 1)$$

We can therefore express the B_l in terms of A_l :

$$(0.9) \quad B_1 = \frac{p}{4\pi\epsilon} + A_1 R^3$$

$$(0.10) \quad B_l = R^{2l+1} A_l \quad (l \neq 1)$$

Using the condition on the derivatives 0.2 and equating coefficients as before, we get

$$(0.11) \quad -\frac{p}{2\pi R^3} + \epsilon A_1 = -\epsilon_0 \frac{2B_1}{R^3}$$

$$(0.12) \quad \epsilon A_l l R^{l-1} = -\epsilon_0 \frac{(l+1)B_l}{R^{l+2}} \quad (l \neq 1)$$

Substituting for B_l into the second equation gives

$$(0.13) \quad \epsilon A_l l R^{2l+1} = -\epsilon_0 (l+1) A_l R^{2l+1}$$

The only way this can be satisfied is if $A_l = 0$ for $l \neq 1$, so we get $A_l = B_l = 0$ for $l \neq 1$.

For the $l = 1$ case, we can solve the two equations in A_1 and B_1 to get (using $\epsilon = \epsilon_0 \epsilon_r$):

$$(0.14) \quad A_1 = \frac{p}{2\pi\epsilon R^3} \left[\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right]$$

$$(0.15) \quad = \frac{2p}{4\pi R^3 \epsilon} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)$$

$$(0.16) \quad B_1 = \frac{p}{4\pi\epsilon} \left[1 + 2 \frac{\epsilon_r - 1}{\epsilon_r + 2} \right]$$

$$(0.17) \quad = \frac{p}{4\pi\epsilon_0} \left(\frac{3}{\epsilon_r + 2} \right)$$

The potential is

$$(0.18) \quad V_{in}(r) = \frac{p \cos \theta}{4\pi\epsilon r^2} + \frac{2pr \cos \theta}{4\pi R^3 \epsilon} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)$$

$$(0.19) \quad = \frac{p \cos \theta}{4\pi\epsilon r^2} \left[1 + 2 \frac{r^3}{R^3} \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right) \right]$$

$$(0.20) \quad V_{out}(r) = \frac{p \cos \theta}{4\pi\epsilon_0 r^2} \left(\frac{3}{\epsilon_r + 2} \right)$$