DIPOLE IN DIELECTRIC SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.34.

Another example of solving a boundary value problem in a system with linear dielectrics. Suppose we have an ideal dipole **p** at the centre of a sphere of dielectric (dielectric constant ε_r) and radius R. What is the potential at any point (inside or outside the sphere)?

We can use a similar approach to that of the problem of a dielectric cylinder in an electric field. We first specify the boundary conditions. At the surface of the sphere, the potential must be continuous, so we have

$$V_{in}(R) = V_{out}(R) \tag{1}$$

As we saw in the cylinder problem, the condition on the normal derivative of the potential is, since on the outside of the sphere, $\varepsilon = \varepsilon_0$:

$$\varepsilon \left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = \varepsilon_0 \left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} \tag{2}$$

Since there is no external field, we must have $V \to 0$ as $r \to \infty$. Finally, as $r \to 0$, the potential must behave like that of an ideal dipole, so, assuming that **p** points in the $+z$ direction:

$$\lim_{r\to 0} V_{in}(r) = \frac{p\cos\theta}{4\pi\varepsilon r^2} \tag{3}$$

Note that we're using ε rather than ε_0 in this formula, since the dipole is inside the dielectric and the potential is reduced by a factor of ε_r, where $\varepsilon = \varepsilon_0\varepsilon_r$.

With these conditions, we must solve Laplace's equation in spherical coordinates, so we can quote the general form of the solution:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{4}$$

Inside the sphere, we have

$$V_{in} = \frac{p\cos\theta}{4\pi\varepsilon r^2} + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{5}$$

Outside, we have

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{6}$$

Applying the continuity condition 1 and equating coefficients of the P_l, we get

$$\frac{p}{4\pi\varepsilon R^2} + A_1 R = \frac{B_1}{R^2} \tag{7}$$

$$A_l R^l = \frac{B_l}{R^{l+1}} \quad (l \neq 1) \tag{8}$$

We can therefore express the B_l in terms of A_l:

$$B_1 = \frac{p}{4\pi\varepsilon} + A_1 R^3 \tag{9}$$

$$B_l = R^{2l+1} A_l \quad (l \neq 1) \tag{10}$$

Using the condition on the derivatives 2 and equating coefficients as before, we get

$$-\frac{p}{2\pi R^3} + \varepsilon A_1 = -\varepsilon_0 \frac{2B_1}{R^3} \tag{11}$$

$$\varepsilon A_l l R^{l-1} = -\varepsilon_0 \frac{(l+1) B_l}{R^{l+2}} \quad (l \neq 1) \tag{12}$$

Substituting for B_l into the second equation gives

$$\varepsilon A_l l R^{2l+1} = -\varepsilon_0 (l+1) A_l R^{2l+1} \tag{13}$$

The only way this can be satisfied is if $A_l = 0$ for $l \neq 1$, so we get $A_l = B_l = 0$ for $l \neq 1$.

For the $l = 1$ case, we can solve the two equations in A_1 and B_1 to get (using $\varepsilon = \varepsilon_0 \varepsilon_r$):

$$A_1 = \frac{p}{2\pi\varepsilon R^3}\left[\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right] \tag{14}$$

$$= \frac{2p}{4\pi R^3\varepsilon}\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right) \tag{15}$$

$$B_1 = \frac{p}{4\pi\varepsilon}\left[1+2\frac{\varepsilon_r-1}{\varepsilon_r+2}\right] \tag{16}$$

$$= \frac{p}{4\pi\varepsilon_0}\left(\frac{3}{\varepsilon_r+2}\right) \tag{17}$$

The potential is

$$V_{in}(r) = \frac{p\cos\theta}{4\pi\varepsilon r^2}+\frac{2pr\cos\theta}{4\pi R^3\varepsilon}\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right) \tag{18}$$

$$= \frac{p\cos\theta}{4\pi\varepsilon r^2}\left[1+2\frac{r^3}{R^3}\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right)\right] \tag{19}$$

$$V_{out}(r) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\left(\frac{3}{\varepsilon_r+2}\right) \tag{20}$$