UNIQUENESS OF POTENTIAL IN DIELECTRICS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.35.

When we considered electric fields in a vacuum, we found that if we specify the charge distribution in some region $\mathcal{V}$, and also specify either the potential or its normal derivative on the boundary of that region, then the potential inside $\mathcal{V}$ is unique. Here, we'll show that uniqueness of the potential also applies to the case where $\mathcal{V}$ contains some linear dielectric. The assumptions we make are:

(1) The potential V is specified on all boundaries of $\mathcal{V}$.
(2) The free charge distribution ρ_f is specified everywhere within $\mathcal{V}$.
(3) The distribution of dielectric within $\mathcal{V}$ is fixed, and all dielectric constants are specified.

The proof follows a similar line of reasoning to that used in the electric field case. As before we'll suppose that there are two distinct potentials V_1 and V_2 that satisfy the conditions. We'll also define the displacements due to these potentials as $\mathbf{D}_1$ and $\mathbf{D}_2$. Now we consider the difference between these two solutions, so we have $V_3 \equiv V_1 - V_2$ and $\mathbf{D}_3 \equiv \mathbf{D}_1 - \mathbf{D}_2$. Now we look at this volume integral and convert it to a surface integral in the usual way:

$$\int_{\mathcal{V}} \nabla\cdot(V_3\mathbf{D}_3)\,d^3\mathbf{r} \quad = \quad \int_A V_3\mathbf{D}_3\cdot d\mathbf{a} \tag{1}$$

By assumption, on the surface A, $V_3 = V_1 - V_2 = 0$ since the potential is specified everywhere on the boundary. Therefore:

$$\int_{\mathcal{V}} \nabla\cdot(V_3\mathbf{D}_3)\,d^3\mathbf{r} = 0 \tag{2}$$

Now we expand the integrand using a standard theorem from vector calculus:

$$\nabla\cdot(V_3\mathbf{D}_3) = \mathbf{D}_3\cdot\nabla V_3 + V_3\nabla\cdot\mathbf{D}_3 \tag{3}$$

From the formula for the divergence of the displacement:

$$\nabla\cdot\mathbf{D}_3 = \nabla\cdot(\mathbf{D}_1-\mathbf{D}_2) \tag{4}$$
$$= \rho_f-\rho_f \tag{5}$$
$$= 0 \tag{6}$$

This follows, since the free charge distribution was fixed by assumption. Therefore we have

$$\nabla\cdot(V_3\mathbf{D}_3) = \mathbf{D}_3\cdot\nabla V_3 \tag{7}$$

For a linear dielectric, $\mathbf{D}=\varepsilon\mathbf{E}$ and in general, $\mathbf{E}=-\nabla V$, so

$$\nabla\cdot(V_3\mathbf{D}_3) = \mathbf{D}_3\cdot\nabla V_3 \tag{8}$$
$$= -\varepsilon E_3^2 \tag{9}$$
$$= -\varepsilon\left|\mathbf{E}_1-\mathbf{E}_2\right|^2 \tag{10}$$

Thus the volume integral becomes

$$\int_{\mathscr{V}}\nabla\cdot(V_3\mathbf{D}_3)\,d^3\mathbf{r} = -\int_{\mathscr{V}}\varepsilon\left|\mathbf{E}_1-\mathbf{E}_2\right|^2 d^3\mathbf{r} \tag{11}$$
$$= 0 \tag{12}$$

(Note that we can't take ε outside the integral since in general it varies over the volume, depending on what dielectrics are present.)

Now the integrand is non-negative everywhere, since the permittivity $\varepsilon\ge\varepsilon_0$ so the only way the integral can be zero is if $\mathbf{E}_1=\mathbf{E}_2$ everywhere inside $\mathscr{V}$. This means that $V_1-V_2=k$ for some constant k, but since $V_1=V_2$ on the boundary, we must have $k=0$ and $V_1=V_2$ everywhere.

PINGBACKS

Pingback: Conducting sphere half-embedded in dielectric plane