UNIQUENESS OF POTENTIAL IN DIELECTRICS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.35.

When we considered electric fields in a vacuum, we found that if we specify the charge distribution in some region $\mathcal{V}$, and also specify either the potential or its normal derivative on the boundary of that region, then the potential inside $\mathcal{V}$ is unique. Here, we'll show that uniqueness of the potential also applies to the case where $\mathcal{V}$ contains some linear dielectric. The assumptions we make are:

(1) The potential V is specified on all boundaries of $\mathcal{V}$.
(2) The free charge distribution ρ_f is specified everywhere within $\mathcal{V}$.
(3) The distribution of dielectric within $\mathcal{V}$ is fixed, and all dielectric constants are specified.

The proof follows a similar line of reasoning to that used in the electric field case. As before we'll suppose that there are two distinct potentials V_1 and V_2 that satisfy the conditions. We'll also define the displacements due to these potentials as $\mathbf{D}_1$ and $\mathbf{D}_2$. Now we consider the difference between these two solutions, so we have $V_3 \equiv V_1 - V_2$ and $\mathbf{D}_3 \equiv \mathbf{D}_1 - \mathbf{D}_2$. Now we look at this volume integral and convert it to a surface integral in the usual way:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{D}_3)\, d^3\mathbf{r} = \int_A V_3 \mathbf{D}_3 \cdot d\mathbf{a} \tag{1}$$

By assumption, on the surface A, $V_3 = V_1 - V_2 = 0$ since the potential is specified everywhere on the boundary. Therefore:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{D}_3)\, d^3\mathbf{r} = 0 \tag{2}$$

Now we expand the integrand using a standard theorem from vector calculus:

$$\nabla \cdot (V_3 \mathbf{D}_3) = \mathbf{D}_3 \cdot \nabla V_3 + V_3 \nabla \cdot \mathbf{D}_3 \tag{3}$$

From the formula for the divergence of the displacement:

$$\nabla\cdot\mathbf{D}_{3} = \nabla\cdot\left(\mathbf{D}_{1}-\mathbf{D}_{2}\right) \quad (4)$$
$$= \rho_{f}-\rho_{f} \quad (5)$$
$$= 0 \quad (6)$$

This follows, since the free charge distribution was fixed by assumption. Therefore we have

$$\nabla\cdot\left(V_{3}\mathbf{D}_{3}\right)=\mathbf{D}_{3}\cdot\nabla V_{3} \quad (7)$$

For a linear dielectric, $\mathbf{D}=\epsilon\mathbf{E}$ and in general, $\mathbf{E}=-\nabla V$, so

$$\nabla\cdot\left(V_{3}\mathbf{D}_{3}\right) = \mathbf{D}_{3}\cdot\nabla V_{3} \quad (8)$$
$$= -\epsilon E_{3}^{2} \quad (9)$$
$$= -\epsilon\left|\mathbf{E}_{1}-\mathbf{E}_{2}\right|^{2} \quad (10)$$

Thus the volume integral becomes

$$\int_{\mathcal{V}}\nabla\cdot\left(V_{3}\mathbf{D}_{3}\right)d^{3}\mathbf{r} = -\int_{\mathcal{V}}\epsilon\left|\mathbf{E}_{1}-\mathbf{E}_{2}\right|^{2}d^{3}\mathbf{r} \quad (11)$$
$$= 0 \quad (12)$$

(Note that we can't take ϵ outside the integral since in general it varies over the volume, depending on what dielectrics are present.)

Now the integrand is non-negative everywhere, since the permittivity $\epsilon\geq\epsilon_{0}$ so the only way the integral can be zero is if $\mathbf{E}_{1}=\mathbf{E}_{2}$ everywhere inside $\mathcal{V}$. This means that $V_{1}-V_{2}=k$ for some constant k, but since $V_{1}=V_{2}$ on the boundary, we must have $k=0$ and $V_{1}=V_{2}$ everywhere.

PINGBACKS

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