

UNIQUENESS OF POTENTIAL IN DIELECTRICS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.35.

When we considered electric fields in a vacuum, we found that if we specify the charge distribution in some region \mathcal{V} , and also specify either the potential or its normal derivative on the boundary of that region, then the potential inside \mathcal{V} is unique. Here, we'll show that uniqueness of the potential also applies to the case where \mathcal{V} contains some linear dielectric. The assumptions we make are:

- (1) The potential V is specified on all boundaries of \mathcal{V} .
- (2) The free charge distribution ρ_f is specified everywhere within \mathcal{V} .
- (3) The distribution of dielectric within \mathcal{V} is fixed, and all dielectric constants are specified.

The proof follows a similar line of reasoning to that used in the electric field case. As before we'll suppose that there are two distinct potentials V_1 and V_2 that satisfy the conditions. We'll also define the displacements due to these potentials as \mathbf{D}_1 and \mathbf{D}_2 . Now we consider the difference between these two solutions, so we have $V_3 \equiv V_1 - V_2$ and $\mathbf{D}_3 \equiv \mathbf{D}_1 - \mathbf{D}_2$. Now we look at this volume integral and convert it to a surface integral in the usual way:

$$(0.1) \quad \int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{D}_3) d^3 \mathbf{r} = \int_A V_3 \mathbf{D}_3 \cdot d\mathbf{a}$$

By assumption, on the surface A , $V_3 = V_1 - V_2 = 0$ since the potential is specified everywhere on the boundary. Therefore:

$$(0.2) \quad \int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{D}_3) d^3 \mathbf{r} = 0$$

Now we expand the integrand using a standard theorem from vector calculus:

$$(0.3) \quad \nabla \cdot (V_3 \mathbf{D}_3) = \mathbf{D}_3 \cdot \nabla V_3 + V_3 \nabla \cdot \mathbf{D}_3$$

From the formula for the divergence of the displacement:

$$(0.4) \quad \nabla \cdot \mathbf{D}_3 = \nabla \cdot (\mathbf{D}_1 - \mathbf{D}_2)$$

$$(0.5) \quad = \rho_f - \rho_f$$

$$(0.6) \quad = 0$$

This follows, since the free charge distribution was fixed by assumption. Therefore we have

$$(0.7) \quad \nabla \cdot (V_3 \mathbf{D}_3) = \mathbf{D}_3 \cdot \nabla V_3$$

For a linear dielectric, $\mathbf{D} = \epsilon \mathbf{E}$ and in general, $\mathbf{E} = -\nabla V$, so

$$(0.8) \quad \nabla \cdot (V_3 \mathbf{D}_3) = \mathbf{D}_3 \cdot \nabla V_3$$

$$(0.9) \quad = -\epsilon E_3^2$$

$$(0.10) \quad = -\epsilon |\mathbf{E}_1 - \mathbf{E}_2|^2$$

Thus the volume integral becomes

$$(0.11) \quad \int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{D}_3) d^3 \mathbf{r} = - \int_{\mathcal{V}} \epsilon |\mathbf{E}_1 - \mathbf{E}_2|^2 d^3 \mathbf{r}$$

$$(0.12) \quad = 0$$

(Note that we can't take ϵ outside the integral since in general it varies over the volume, depending on what dielectrics are present.)

Now the integrand is non-negative everywhere, since the permittivity $\epsilon \geq \epsilon_0$ so the only way the integral can be zero is if $\mathbf{E}_1 = \mathbf{E}_2$ everywhere inside \mathcal{V} . This means that $V_1 - V_2 = k$ for some constant k , but since $V_1 = V_2$ on the boundary, we must have $k = 0$ and $V_1 = V_2$ everywhere.

PINGBACKS

Pingback: Conducting sphere half-embedded in dielectric plane