CONDUCTING SPHERE HALF-EMBEDDED IN DIELECTRIC PLANE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 4.36.

An interesting special case of a dielectric problem is one in which a conducting sphere is partially embedded in an infinite half-space of dielectric. That is, we have a metal sphere of radius R with its centre at the origin, and the space $z<0$ is completely filled with a linear dielectric with susceptibility χ_e, so that the lower hemisphere of the conductor is embedded in the dielectric with the top hemisphere in vacuum.

If we hold the sphere at a potential of V_0 then in the absence of the dielectric, the potential would be

$$V(r)=\begin{cases}V_0 & r<R\\ V_0\frac{R}{r} & r>R\end{cases} \tag{1}$$

From this we get the field (which is zero inside the sphere of course):

$$\mathbf{E} = -\nabla V \tag{2}$$

$$= \frac{V_0 R}{r^2}\hat{\mathbf{r}} \quad (r>R) \tag{3}$$

For a linear dielectric, the polarization is (for $z<0$):

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \tag{4}$$

$$= \varepsilon_0\chi_e\frac{V_0 R}{r^2}\hat{\mathbf{r}} \tag{5}$$

The volume bound charge induced by this is

$$\rho_b = -\nabla\cdot\mathbf{P} \tag{6}$$

$$= 0 \tag{7}$$

(since $\nabla\cdot\left(\hat{\mathbf{r}}/r^2\right)=4\pi\delta_3(\mathbf{r})$ and we are considering only points with $r>R$ so we avoid the singularity at the origin).

The surface bound charge is $\sigma_b=\mathbf{P}\cdot\hat{\mathbf{n}}$ and is zero on the plane $z=0$ since $\mathbf{P}\perp\hat{\mathbf{n}}$. That is, since $\mathbf{P}$ is parallel to $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}$ is horizontal across the plane $z=0$, all the polarization is parallel to the plane so there is no bound surface charge.

On the lower hemisphere, we consider the surface of the dielectric which has a surface normal pointing towards the origin, which means that $\hat{\mathbf{n}}=-\hat{\mathbf{r}}$ so on this surface

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \tag{8}$$

$$= -\epsilon_0\chi_e\frac{V_0}{R} \tag{9}$$

In order for the sphere to be maintained at a constant potential, the surface charge density on the lower hemisphere would have to increase to compensate for this induced bound charge on the dielectric. This is the same argument as that used when we increase the capacitance of a capacitor by putting dielectric between its plates. Because the dielectric cancels out some of the field between the plates, we need to put more charge on the plates in order to maintain the same potential difference between the plates. In this case, we're trying to maintain the same potential difference between the sphere and infinity (where $V=0$), so we need to increase the charge on the lower hemisphere to maintain it at V_0.

In the absence of dielectric, we can work out the surface charge density on the sphere from the field, since across a surface layer of charge

$$\Delta\mathbf{E}_\perp = \frac{\sigma}{\epsilon_0} \tag{10}$$

Since $\mathbf{E}=0$ inside and $\mathbf{E}=\frac{V_0}{R}\hat{\mathbf{r}}$ outside, we have for the upper hemisphere

$$\sigma_\uparrow = \frac{\epsilon_0 V_0}{R} \tag{11}$$

In the lower hemisphere, the net surface density must be the same, so

$$\sigma_\downarrow = \frac{\epsilon_0 V_0}{R}+\epsilon_0\chi_e\frac{V_0}{R} \tag{12}$$

$$= \epsilon\frac{V_0}{R} \tag{13}$$

where $\epsilon=\epsilon_0(1+\chi_e)=\epsilon_0\epsilon_r$. That is, the net charge density is the same as if the dielectric was not present at all. Thus by the uniqueness of the potential in systems containing dielectrics, the potential for the half-embedded sphere is the same as that for a sphere without any dielectric.

This argument hinges on the fact that the boundary of the dielectric lies on a surface that is parallel to the field (the plane $z = 0$ here). Thus we could use the same argument for a sphere embedded in any dielectric in which the surface of the dielectric lay entirely parallel to the field, for example a cone centred at the origin. It would *not* work for, say, a sphere embedded in a dielectric plane that didn't split the sphere into two equal hemispheres.