CYCLOID MOTION IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.2.

One of the classic examples of how mixing electric and magnetic fields can produce unusual results is the case of cycloid motion. The idea is that we have a constant magnetic field pointing in the x direction, and a constant electric field perpendicular to the magnetic field. We might as well align the electric field with the z axis.

If we place a particle with charge q at the origin and either give it an initial velocity perpendicular to $\mathbf{B}$ or start it at rest, then neither the electric nor the magnetic field will exert a force in the x direction, so the motion is restricted entirely to the yz plane.

Combining the Lorentz force law $\mathbf{F}=q\mathbf{v}\times\mathbf{B}$ and electrostatics with Newton's law $\mathbf{F}=m\mathbf{a}$ we get, using a dot to denote a derivative w.r.t. time:

$$m\ddot{y} = qB\dot{z} \tag{1}$$

$$m\ddot{z} = qE-qB\dot{y} \tag{2}$$

At this point, we can note that it's quite easy to generalize this system so that $\mathbf{E}$ has an arbitrary direction perpendicular to $\mathbf{B}$. However, there's not much point in doing so, since this amounts merely to a rotation of the coordinates about the x axis. Giving $\mathbf{E}$ a component along x, however, *does* introduce significant complexity into the system, since we then have a force component along x, so we need another differential equation in the above set.

We can solve the system of 2 ODEs above by using software (or by hand, if you're familiar with methods of solving systems like this), and we get

$$y(t) = -\frac{c_3}{\omega}\cos\omega t+\frac{c_2}{\omega}\sin\omega t+\frac{E}{B}t+c_4 \tag{3}$$

$$z(t) = \frac{c_3}{\omega}\sin\omega t+\frac{c_2}{\omega}\cos\omega t+\frac{E}{\omega B}+c_1 \tag{4}$$

where $\omega\equiv\frac{qB}{m}$ and the c_is are constants of integration. Note that ω has units of inverse time, as we can see from the Lorentz force law above. qB

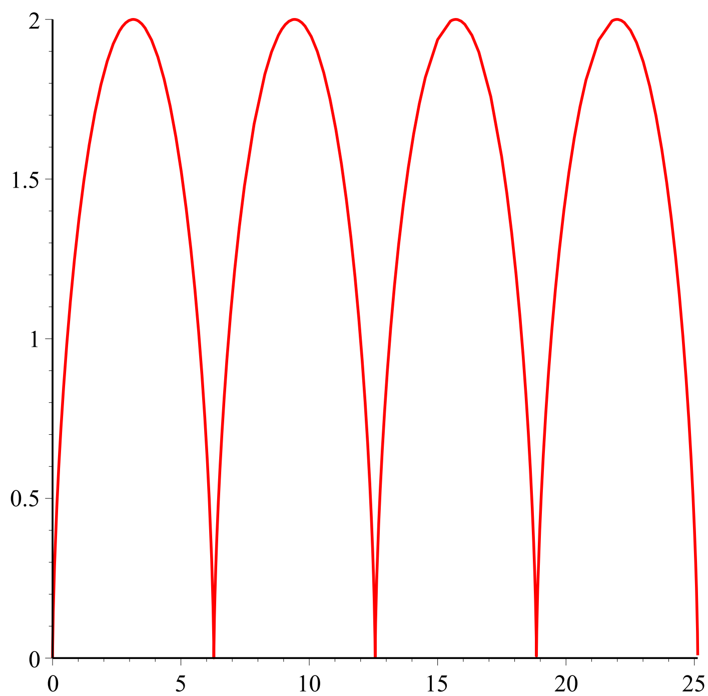
has units of force/velocity, which has units of mass $\times$ time^{-1}. (For those reading Griffiths's book, the constants in our solution can be redefined to produce Griffiths's equation 5.6, since E,B,q,m and ω are all constants here.)

The four constants can be determined by specifying the initial position and velocity of the particle. If it starts at rest at the origin (that is, $x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0$), we can apply these conditions to the solution (again, using software to ease the mathematics), and get

$$y(t) = -\frac{E}{\omega B}\sin\omega t + \frac{E}{B}t \quad (5)$$

$$z(t) = -\frac{E}{\omega B}\cos\omega t + \frac{E}{\omega B} \quad (6)$$

These two equations give the trajectory in parametric form, so we can vary t to get a plot of z versus y (we've taken all the constants to be 1 $E = B = q = m = 1$ for convenience):



Taking an initial velocity of $\frac{E}{B}$ along the y axis ($\dot{y}(0) = \frac{E}{B}$), we get

$$y(t) = \frac{E}{B}t \quad (7)$$

$$z(t) = 0 \quad (8)$$

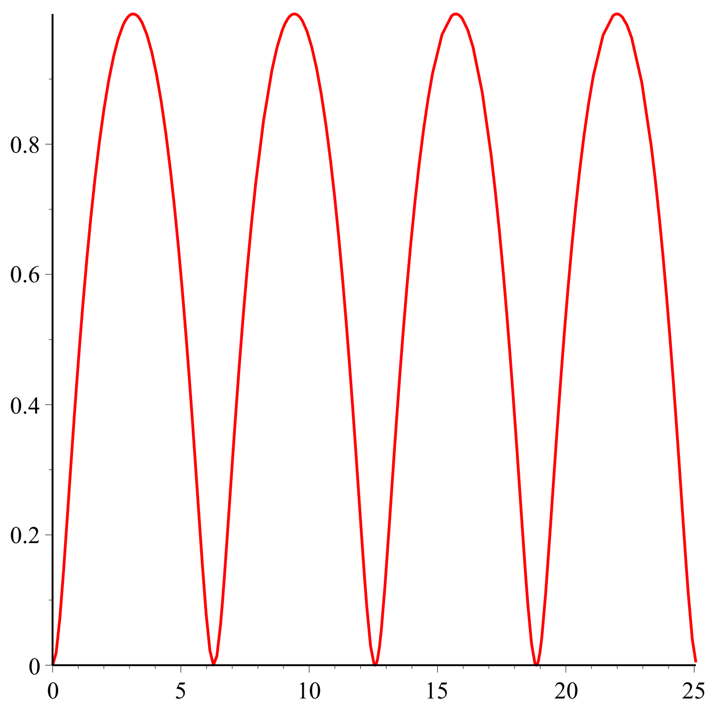
In this case the particle moves at constant speed along the y axis. Why? The electric force is qE in the $+z$ direction (assuming q is positive), and the initial magnetic force is $qvB = q\frac{E}{B}B = qE$ in the $-z$ direction (use the right hand rule in the cross product). Thus the two forces cancel each other, so there is no net acceleration and the initial velocity remains unchanged.

If $\dot{y}(0) = \frac{E}{2B}$, we get

$$y(t) = -\frac{E}{2\omega B}\sin\omega t + \frac{E}{B}t \quad (9)$$

$$z(t) = -\frac{E}{2\omega B}\cos\omega t + \frac{E}{2\omega B} \quad (10)$$

This gives a similar trajectory to the first example, but with a reduced amplitude:

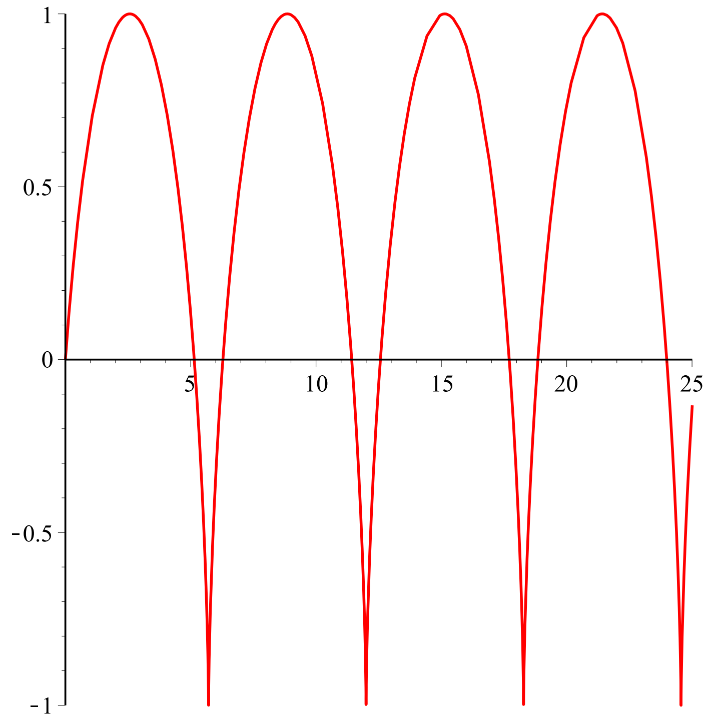


Finally, if we start the particle off travelling at an angle of $\pi/4$ w.r.t. the y axis, so that $\dot{y}(0) = \dot{z}(0) = \frac{E}{B}$, we get

$$y(t) = -\frac{E}{\omega B}\cos\omega t + \frac{E}{B}t + \frac{E}{\omega B} \quad (11)$$

$$z(t) = \frac{E}{\omega B}\sin\omega t \quad (12)$$

This gives a trajectory identical in shape to the first case, but shifted down so that its midpoint is now on the y axis:



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