

VOLUME CURRENT DENSITY AND DIPOLE MOMENT

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.7.

There is a relation between the volume current density and the dipole moment. Suppose we have a collection of charges and currents contained within a finite volume. Consider the integral

$$\int_V \nabla \cdot (x\mathbf{J}) d^3\mathbf{r} \quad (1)$$

Looking at the integrand, we have

$$\nabla \cdot (x\mathbf{J}) = \nabla x \cdot \mathbf{J} + x \nabla \cdot \mathbf{J} \quad (2)$$

$$= \hat{\mathbf{x}} \cdot \mathbf{J} + x \nabla \cdot \mathbf{J} \quad (3)$$

$$= J_x + x \nabla \cdot \mathbf{J} \quad (4)$$

The current density represents the rate of flow of charge across a surface, so the total rate of change of charge must be equal to the net flow across the surface. That is

$$\int_A \mathbf{J} \cdot d\mathbf{a} = -\frac{d}{dt} \int_V \rho d^3\mathbf{r} \quad (5)$$

where the minus sign indicates that a net flow of charge out of the volume (indicated by the integral on the LHS being positive) means a decrease in the amount of charge inside the volume.

By the divergence theorem, the integral on the LHS can be converted into a volume integral, so we get

$$\int_A \mathbf{J} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{J} d^3\mathbf{r} \quad (6)$$

$$= -\int_V \frac{\partial \rho}{\partial t} d^3\mathbf{r} \quad (7)$$

where we've taken the time derivative inside the integral, and converted it to a partial derivative since ρ depends on space as well.

This relation has to be true for any volume, so we can equate the integrands to get

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad (8)$$

Returning to the first equation, we can now say:

$$\nabla \cdot (x\mathbf{J}) = J_x + x\nabla \cdot \mathbf{J} \quad (9)$$

$$= J_x - x\frac{\partial \rho}{\partial t} \quad (10)$$

Integrating the LHS and using the divergence theorem in the opposite direction, we get

$$\int_V \nabla \cdot (x\mathbf{J}) d^3\mathbf{r} = \int_A x\mathbf{J} \cdot d\mathbf{a} \quad (11)$$

Now if we take the bounding surface A to be outside all the charges and currents, then the RHS is zero, so we get

$$\int_V J_x d^3\mathbf{r} = \int_V x\frac{\partial \rho}{\partial t} d^3\mathbf{r} \quad (12)$$

Clearly we can write similar equations for the y and z components and then multiply each equation by the corresponding unit vector and add all three up to get

$$\int_V \mathbf{J} d^3\mathbf{r} = \int_V \mathbf{r} \frac{\partial \rho}{\partial t} d^3\mathbf{r} \quad (13)$$

The integral on the RHS is

$$\int_V \mathbf{r} \frac{\partial \rho}{\partial t} d^3\mathbf{r} = \frac{d\mathbf{p}}{dt} \quad (14)$$

where \mathbf{p} is the total dipole moment of the charge in the volume. We therefore get the relation between current density and dipole moment:

$$\int_V \mathbf{J} d^3\mathbf{r} = \frac{d\mathbf{p}}{dt} \quad (15)$$

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