VOLUME CURRENT DENSITY AND DIPOLE MOMENT

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.7.

There is a relation between the volume current density and the dipole moment. Suppose we have a collection of charges and currents contained within a finite volume. Consider the integral

$$\int_V \nabla\cdot(x\mathbf{J})\,d^3\mathbf{r} \tag{1}$$

Looking at the integrand, we have

$$\nabla\cdot(x\mathbf{J}) = \nabla x\cdot\mathbf{J}+x\nabla\cdot\mathbf{J} \tag{2}$$
$$= \hat{\mathbf{x}}\cdot\mathbf{J}+x\nabla\cdot\mathbf{J} \tag{3}$$
$$= J_x+x\nabla\cdot\mathbf{J} \tag{4}$$

The current density represents the rate of flow of charge across a surface, so the total rate of change of charge must be equal to the net flow across the surface. That is

$$\int_A \mathbf{J}\cdot d\mathbf{a} = -\frac{d}{dt}\int_V \rho d^3\mathbf{r} \tag{5}$$

where the minus sign indicates that a net flow of charge out of the volume (indicated by the integral on the LHS being positive) means a decrease in the amount of charge inside the volume.

By the divergence theorem, the integral on the LHS can be converted into a volume integral, so we get

$$\int_A \mathbf{J}\cdot d\mathbf{a} = \int_V \nabla\cdot\mathbf{J}d^3\mathbf{r} \tag{6}$$
$$= -\int_V \frac{\partial\rho}{\partial t}d^3\mathbf{r} \tag{7}$$

where we've taken the time derivative inside the integral, and converted it to a partial derivative since ρ depends on space as well.

This relation has to be true for any volume, so we can equate the integrands to get

$$\nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t} \tag{8}$$

Returning to the first equation, we can now say:

$$\begin{aligned}\nabla\cdot(x\mathbf{J}) &= J_x+x\nabla\cdot\mathbf{J} &(9)\\ &= J_x-x\frac{\partial\rho}{\partial t} &(10)\end{aligned}$$

Integrating the LHS and using the divergence theorem in the opposite direction, we get

$$\int_V\nabla\cdot(x\mathbf{J})\,d^3\mathbf{r}=\int_A x\mathbf{J}\cdot d\mathbf{a} \tag{11}$$

Now if we take the bounding surface A to be outside all the charges and currents, then the RHS is zero, so we get

$$\int_V J_x d^3\mathbf{r}=\int_V x\frac{\partial\rho}{\partial t}d^3\mathbf{r} \tag{12}$$

Clearly we can write similar equations for the y and z components and then multiply each equation by the corresponding unit vector and add all three up to get

$$\int_V \mathbf{J}d^3\mathbf{r}=\int_V \mathbf{r}\frac{\partial\rho}{\partial t}d^3\mathbf{r} \tag{13}$$

The integral on the RHS is

$$\int_V \mathbf{r}\frac{\partial\rho}{\partial t}d^3\mathbf{r}=\frac{d\mathbf{p}}{dt} \tag{14}$$

where $\mathbf{p}$ is the total dipole moment of the charge in the volume. We therefore get the relation between current density and dipole moment:

$$\int_V \mathbf{J}d^3\mathbf{r}=\frac{d\mathbf{p}}{dt} \tag{15}$$

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