BIOT-SAVART LAW - CURRENT LOOPS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.8.

As mentioned earlier, besides feeling a force from an external magnetic field, an electric current also produces its own magnetic field. The experimentally determined rule for calculating this generated magnetic field is known as the Biot-Savart law. For a steady current (one that doesn't vary with time) in a wire, this law can be written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dl' \tag{1}$$

where $\mathbf{r}'$ is a location on the wire and $\mathbf{r}$ is the point at which you want to determine the magnetic field. The constant μ_0 is known as the permeability of free space, and is the magnetic analogue to ϵ_0 in electrostatics. Its value is

$$\mu_0 = 1.25663706\times10^{-6}\text{m kg s}^{-2}\text{Amp}^{-2} \tag{2}$$

Again, this law isn't derived from anything more fundamental; it's a generalization of experiment, although the value of μ_0 is fixed at exactly $4\pi\times10^{-7}$m kg s^{-2}Amp^{-2}.

As an example, suppose we want to find the field generated by a steady current travelling round a square loop of side length $2R$, at the centre of the square. We can do this by finding the field generated by a single wire segment first.

Because of the cross product in the integrand, the field will be perpendicular to the plane of the square, so if we call this direction the z axis, and set the edge of the square at $x = R$, we can write the integral as (setting $\mathbf{r} = 0$ since we're interested in the field at the origin):

$$\begin{aligned}
\mathbf{B}_1(\mathbf{r}) &= \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\int\frac{|\mathbf{r}-\mathbf{r}'|\sin\theta}{|\mathbf{r}-\mathbf{r}'|^3}dl' &(3)\\
&= \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\int_{-R}^{R}\frac{R}{(R^2+y^2)^{3/2}}dy &(4)\\
&= \hat{\mathbf{z}}\frac{\sqrt{2}I\mu_0}{4\pi R} &(5)
\end{aligned}$$

where θ is the angle between $\mathbf{I}$ and $\mathbf{r}-\mathbf{r}'$, so that $\sin\theta = R/|\mathbf{r}-\mathbf{r}'|$.

By symmetry, the contribution from all 4 sides is equal, so we get for the total field

$$\mathbf{B} = 4\mathbf{B}_1 = \hat{\mathbf{z}}\frac{\sqrt{2}I\mu_0}{\pi R} \tag{6}$$

Now suppose we have a current loop consisting of a regular polygon with n sides. In this case, each side subtends an angle of $2\pi/n$, so if we align one side parallel to the y axis at $x=R$, this side will extend from an angle of $-\pi/n$ to $+\pi/n$, and will have a length of $2R\tan\frac{\pi}{n}$. Now the integral for a single side is

$$\begin{aligned}
\mathbf{B}_1(0) &= \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\int_{-R\tan\frac{\pi}{n}}^{R\tan\frac{\pi}{n}}\frac{R}{(R^2+y^2)^{3/2}}dy &(7)\\
&= \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\frac{2R\tan\frac{\pi}{n}}{R\sqrt{R^2+\left(R\tan\frac{\pi}{n}\right)^2}} &(8)\\
&= \hat{\mathbf{z}}\frac{I\mu_0}{2\pi R}\sin\frac{\pi}{n} &(9)
\end{aligned}$$

Each of the n sides will still contribute an equal amount, so the total field is

$$\mathbf{B} = n\mathbf{B}_1 = \hat{\mathbf{z}}\frac{nI\mu_0}{2\pi R}\sin\frac{\pi}{n} \tag{10}$$

As $n\to\infty$, this formula should give us the field due to a circular loop. In this limit, we can approximate the sine by the first term in its Taylor expansion $\sin\frac{\pi}{n}\approx\frac{\pi}{n}$ so we get

$$\lim_{n\to\infty}\mathbf{B} = \hat{\mathbf{z}}\frac{I\mu_0}{2R} \tag{11}$$

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