# MAGNETIC FIELD OF A SOLENOID

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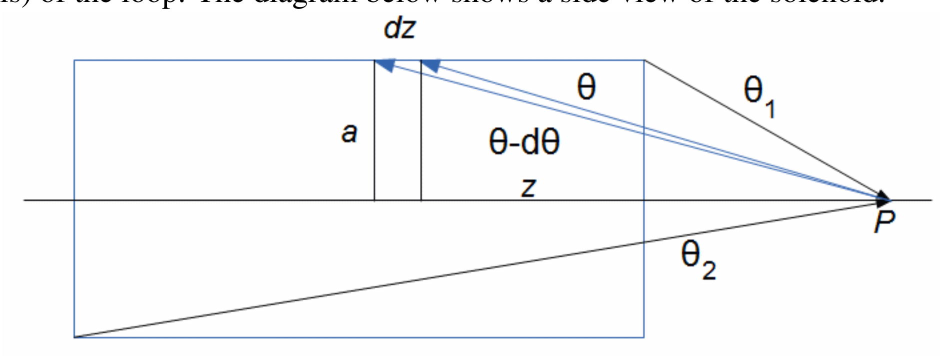
Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.11.

A solenoid is a helical coil of wire wound round an insulating cylinder. We can find the magnetic field due to a solenoid carrying a steady current $I$ as follows. First, we work out the field due to a single circular loop of radius $a$ as measured at a point $P$ on the axis (which we'll take to be the $z$ axis) of the loop. The diagram below shows a side view of the solenoid.




In the diagram, we'll take the circular loop to be the first loop in the solenoid (closest to $P$), whose radius subtends an angle $\theta_1$ at $P$. The distance from $P$ along the axis to the centre of the circle is $z$. (We'll refer to the other parts of the diagram later.)

By symmetry, the components of the field in the $x$ and $y$ directions cancel, so we need calculate only the $z$ component. The Biot-Savart law in this case is

$$\mathbf{B}(\mathbf{r}) = \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\int\frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{0.1}$$

The line element $d\mathbf{l}$ is tangent to the circle, and the vector $\mathbf{r}-\mathbf{r}'$ is always perpendicular to it. Look at the point on the loop where the tail of the$\theta_1$ vector meets the current loop, and assume the current is coming out of the page at this point. Then applying the right-hand rule, we see that the field vector lies in the plane of the figure, and points diagonally to the upper right, making an angle of $\frac{\pi}{2}-\theta_1$ with the $z$ axis. To project out the $z$ component of the field, we multiply by $\cos\left(\frac{\pi}{2}-\theta_1\right) = \sin\theta_1 = \frac{a}{|\mathbf{r}-\mathbf{r}'|}$. This angle is a constant, as is $|\mathbf{r}-\mathbf{r}'|$, and the integration extends round the circumference of the circle, so we get

$$B_z = \frac{I\mu_0}{4\pi}\frac{a}{|\mathbf{r}-\mathbf{r}'|}\frac{2\pi a}{|\mathbf{r}-\mathbf{r}'|^2} \tag{0.2}$$

$$= \frac{I\mu_0}{2a}\sin^3\theta_1 \tag{0.3}$$

Now suppose the solenoid has $n$ turns per unit length, and that this number is large enough that we can approximate the solenoid by a circular surface current density of $In$ per unit length, and thus use integration to determine the overall field. To do the integral, we need to work out relation between a change in $\theta$ and a change in $z$, as shown in the diagram. An increase of $dz$ means a decrease in angle of $-d\theta$ as shown. To work out the relation:

$$\sin\theta = \frac{a}{\sqrt{a^2+z^2}} \tag{0.4}$$

$$\sin(\theta-d\theta) = \frac{a}{\sqrt{a^2+(z+dz)^2}} \tag{0.5}$$

We can expand the second equation in a Taylor series and retain only first terms in the differentials, and we get

$$\sin\theta - \cos\theta d\theta = \frac{a}{\sqrt{a^2+z^2}} - \frac{az}{(a^2+z^2)^{3/2}}dz \tag{0.6}$$

Replacing the terms on the right by trig functions, we get

$$\frac{a}{\sqrt{a^2+z^2}} - \frac{az}{(a^2+z^2)^{3/2}}dz = \sin\theta - \frac{1}{a}\cos\theta\sin^2\theta dz \tag{0.7}$$

Thus we get

$$dz = \frac{a}{\sin^2\theta}d\theta \tag{0.8}$$

The current in an infinitesimal slice of the solenoid is therefore

$$Indz = \frac{Ina}{\sin^2\theta}d\theta \tag{0.9}$$

We can now find the total field by doing the integral

$$B_z = \int_{\theta_2}^{\theta_1} \frac{\mu_0}{4\pi a}\sin^3\theta_1 \frac{Ina}{\sin^2\theta} d\theta \quad (0.10)$$

$$= \frac{n\mu_0 I}{2}\left(\cos\theta_2 - \cos\theta_1\right) \quad (0.11)$$

This might look like it's independent of the radius $a$, but this dependence is included in the angles.

For an infinite solenoid, $\theta_1 \to \pi$ and $\theta_2 \to 0$ and we get

$$B_\infty = n\mu_0 I \quad (0.12)$$

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