

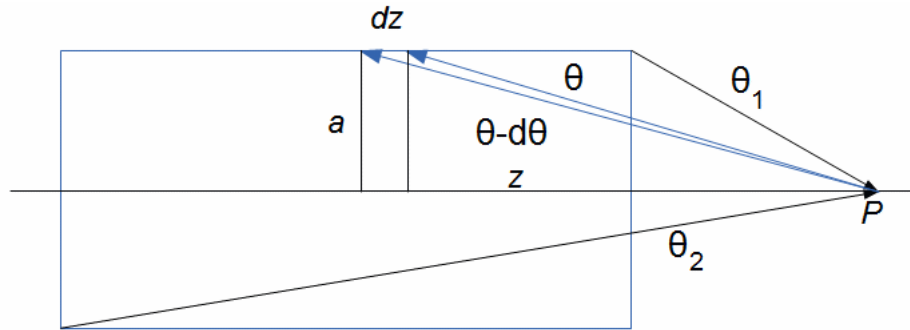
## MAGNETIC FIELD OF A SOLENOID

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.11.

A solenoid is a helical coil of wire wound round an insulating cylinder. We can find the magnetic field due to a solenoid carrying a steady current  $I$  as follows. First, we work out the field due to a single circular loop of radius  $a$  as measured at a point  $P$  on the axis (which we'll take to be the  $z$  axis) of the loop. The diagram below shows a side view of the solenoid.



In the diagram, we'll take the circular loop to be the first loop in the solenoid (closest to  $P$ ), whose radius subtends an angle  $\theta_1$  at  $P$ . The distance from  $P$  along the axis to the centre of the circle is  $z$ . (We'll refer to the other parts of the diagram later.)

By symmetry, the components of the field in the  $x$  and  $y$  directions cancel, so we need calculate only the  $z$  component. The Biot-Savart law in this case is

$$(1) \quad \mathbf{B}(\mathbf{r}) = \hat{\mathbf{z}} \frac{I\mu_0}{4\pi} \int \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

The line element  $d\mathbf{l}$  is tangent to the circle, and the vector  $\mathbf{r} - \mathbf{r}'$  is always perpendicular to it. Look at the point on the loop where the tail of the  $\theta_1$  vector meets the current loop, and assume the current is coming out of the page at this point. Then applying the right-hand rule, we see that the field vector lies in the plane of the figure, and points diagonally to the upper right, making an angle of  $\frac{\pi}{2} - \theta_1$  with the  $z$  axis. To project out the  $z$  component of the field, we multiply by  $\cos\left(\frac{\pi}{2} - \theta_1\right) = \sin \theta_1 = \frac{a}{|\mathbf{r} - \mathbf{r}'|}$ . This angle is a

constant, as is  $|\mathbf{r} - \mathbf{r}'|$ , and the integration extends round the circumference of the circle, so we get

$$(2) \quad B_z = \frac{I\mu_0}{4\pi} \frac{a}{|\mathbf{r} - \mathbf{r}'|} \frac{2\pi a}{|\mathbf{r} - \mathbf{r}'|^2}$$

$$(3) \quad = \frac{I\mu_0}{2a} \sin^3 \theta_1$$

Now suppose the solenoid has  $n$  turns per unit length, and that this number is large enough that we can approximate the solenoid by a circular surface current density of  $In$  per unit length, and thus use integration to determine the overall field. To do the integral, we need to work out relation between a change in  $\theta$  and a change in  $z$ , as shown in the diagram. An increase of  $dz$  means a decrease in angle of  $-d\theta$  as shown. To work out the relation:

$$(4) \quad \sin \theta = \frac{a}{\sqrt{a^2 + z^2}}$$

$$(5) \quad \sin(\theta - d\theta) = \frac{a}{\sqrt{a^2 + (z + dz)^2}}$$

We can expand the second equation in a Taylor series and retain only first terms in the differentials, and we get

$$(6) \quad \sin \theta - \cos \theta d\theta = \frac{a}{\sqrt{a^2 + z^2}} - \frac{az}{(a^2 + z^2)^{3/2}} dz$$

Replacing the terms on the right by trig functions, we get

$$(7) \quad \frac{a}{\sqrt{a^2 + z^2}} - \frac{az}{(a^2 + z^2)^{3/2}} dz = \sin \theta - \frac{1}{a} \cos \theta \sin^2 \theta dz$$

Thus we get

$$(8) \quad dz = \frac{a}{\sin^2 \theta} d\theta$$

The current in an infinitesimal slice of the solenoid is therefore

$$(9) \quad Indz = \frac{Ina}{\sin^2 \theta} d\theta$$

We can now find the total field by doing the integral

$$(10) \quad B_z = \int_{\theta_2}^{\theta_1} \frac{\mu_0}{4\pi a} \sin^3 \theta \frac{Ina}{\sin^2 \theta} d\theta$$

$$(11) \quad = \frac{n\mu_0 I}{2} (\cos \theta_2 - \cos \theta_1)$$

This might look like it's independent of the radius  $a$ , but this dependence is included in the angles.

For an infinite solenoid,  $\theta_1 \rightarrow \pi$  and  $\theta_2 \rightarrow 0$  and we get

$$(12) \quad B_\infty = n\mu_0 I$$

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