MAGNETIC FIELD OF A SOLENOID

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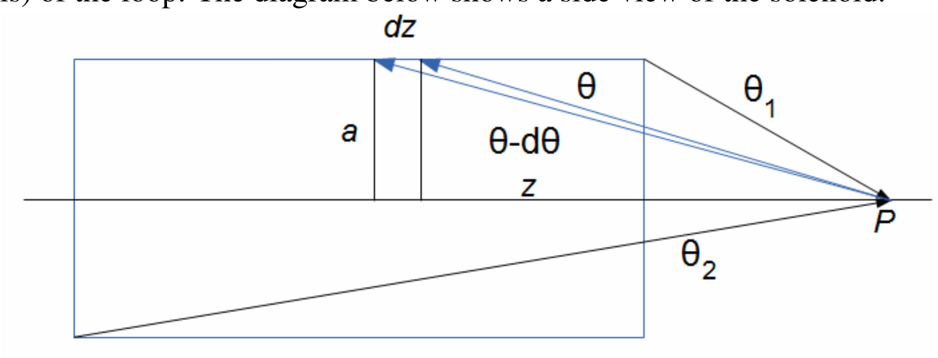
Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.11.

A solenoid is a helical coil of wire wound round an insulating cylinder. We can find the magnetic field due to a solenoid carrying a steady current I as follows. First, we work out the field due to a single circular loop of radius a as measured at a point P on the axis (which we'll take to be the z axis) of the loop. The diagram below shows a side view of the solenoid.



In the diagram, we'll take the circular loop to be the first loop in the solenoid (closest to P), whose radius subtends an angle θ_1 at P. The distance from P along the axis to the centre of the circle is z. (We'll refer to the other parts of the diagram later.)

By symmetry, the components of the field in the x and y directions cancel, so we need calculate only the z component. The Biot-Savart law in this case is

$$\mathbf{B}(\mathbf{r}) = \hat{\mathbf{z}}\frac{I\mu_0}{4\pi}\int\frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1}$$

The line element $d\mathbf{l}$ is tangent to the circle, and the vector $\mathbf{r}-\mathbf{r}'$ is always perpendicular to it. Look at the point on the loop where the tail of theθ_1 vector meets the current loop, and assume the current is coming out of the page at this point. Then applying the right-hand rule, we see that the field vector lies in the plane of the figure, and points diagonally to the upper right, making an angle of $\frac{\pi}{2}-\theta_1$ with the z axis. To project out the z component of the field, we multiply by $\cos\left(\frac{\pi}{2}-\theta_1\right) = \sin\theta_1 = \frac{a}{|\mathbf{r}-\mathbf{r}'|}$. This angle is a constant, as is $|\mathbf{r}-\mathbf{r}'|$, and the integration extends round the circumference of the circle, so we get

$$\begin{aligned} B_z &= \frac{I\mu_0}{4\pi}\frac{a}{|\mathbf{r}-\mathbf{r}'|}\frac{2\pi a}{|\mathbf{r}-\mathbf{r}'|^2} && (2)\\ &= \frac{I\mu_0}{2a}\sin^3\theta_1 && (3)\end{aligned}$$

Now suppose the solenoid has n turns per unit length, and that this number is large enough that we can approximate the solenoid by a circular surface current density of In per unit length, and thus use integration to determine the overall field. To do the integral, we need to work out relation between a change in θ and a change in z, as shown in the diagram. An increase of dz means a decrease in angle of $-d\theta$ as shown. To work out the relation:

$$\begin{aligned} \sin\theta &= \frac{a}{\sqrt{a^2+z^2}} && (4)\\ \sin(\theta-d\theta) &= \frac{a}{\sqrt{a^2+(z+dz)^2}} && (5)\end{aligned}$$

We can expand the second equation in a Taylor series and retain only first terms in the differentials, and we get

$$\sin\theta-\cos\theta d\theta=\frac{a}{\sqrt{a^2+z^2}}-\frac{az}{(a^2+z^2)^{3/2}}dz \tag{6}$$

Replacing the terms on the right by trig functions, we get

$$\frac{a}{\sqrt{a^2+z^2}}-\frac{az}{(a^2+z^2)^{3/2}}dz=\sin\theta-\frac{1}{a}\cos\theta\sin^2\theta dz \tag{7}$$

Thus we get

$$dz=\frac{a}{\sin^2\theta}d\theta \tag{8}$$

The current in an infinitesimal slice of the solenoid is therefore

$$Indz=\frac{Ina}{\sin^2\theta}d\theta \tag{9}$$

We can now find the total field by doing the integral

$$B_z = \int_{\theta_2}^{\theta_1} \frac{\mu_0}{4\pi a} \sin^3\theta_1 \frac{Ina}{\sin^2\theta} d\theta \tag{10}$$

$$= \frac{n\mu_0 I}{2}(\cos\theta_2 - \cos\theta_1) \tag{11}$$

This might look like it's independent of the radius a, but this dependence is included in the angles.

For an infinite solenoid, $\theta_1 \to \pi$ and $\theta_2 \to 0$ and we get

$$B_\infty = n\mu_0 I \tag{12}$$

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