AMPÈRE'S LAW FOR STEADY CURRENTS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.13.

The Biot-Savart law is the mathematical statement of what is observed from experiments on steady currents and their production of magnetic fields. It is, if you like, the magnetic analogue to Coulomb's law in electrostatics. Just as we derived Gauss's law from Coulomb's law and the principle of superposition, so we can derive a law from the Biot-Savart law and the magnetic principle of superposition. This law is known as Ampère's law.

Starting with the most general form of the Biot-Savart law, which treats currents throughout a volume, we have

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{1}$$

where the volume V is assumed to be the finite volume enclosing all currents under consideration.

We can take the curl of this expression if we remember that the derivatives within the curl are with respect to the components of $\mathbf{r}$, *not* $\mathbf{r}'$. That is

$$\nabla_r\times\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V\nabla_r\times\left[\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right]d^3\mathbf{r}' \tag{2}$$

where we've added a suffix r to ∇ to emphasize that it applies only to $\mathbf{r}$ components.

An identity from vector calculus comes in handy here:

$$\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A}) \tag{3}$$

If we set

$$\mathbf{A} = \mathbf{J}\left(\mathbf{r}'\right) \tag{4}$$

$$\mathbf{B} = \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{5}$$

then all terms involving the derivative of $\mathbf{J}$ are zero since derivatives are being taken with respect only to *unprimed* coordinates. Therefore we get

(6)
$$\nabla_r \times \left[\frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right] = -\left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla\right)\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} + \mathbf{J}\left(\mathbf{r}'\right)\left(\nabla\cdot\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

The divergence in the second term can be written in terms of the 3-d delta function. Again, remember that we're taking the derivative only with respect to $\mathbf{r}$ components, so the presence of the $\mathbf{r}'$ serves merely to shift the origin, so we have

$$\nabla\cdot\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = 4\pi\delta_3\left(\mathbf{r}-\mathbf{r}'\right) \tag{7}$$

To handle the first term, we can try to convert it from a volume integral to a surface integral and then do the usual trick of letting the surface go to infinity, and use the assumption that the volume containing the currents is finite to set the integral to zero. To do this, we can split this term into its 3 components and treat each one separately. So for the x component, we have

$$-\left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla\right)\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{8}$$

We need another vector calculus identity, and this time we can use

$$-\mathbf{A}\cdot\nabla f = f\nabla\cdot\mathbf{A} - \nabla\cdot(f\mathbf{A}) \tag{9}$$

There is one slight snag, however. In order to use the divergence theorem to convert a volume integral of a divergence into a surface integral of the normal component of a vector field, the divergence needs to be with respect to the integration variable, which the ∇ operator here *isn't*. However, looking at the term 8, we see that the primed and unprimed coordinates occur in an anti-symmetric fashion (wherever an unprimed variable occurs, it is paired with a primed variable with the opposite sign), so a derivative with respect to an unprimed variable is just the negative of the derivative with respect to a primed variable. That is

$$-\left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla_r\right)\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3} = \left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla_{r'}\right)\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{10}$$

We can now convert the term into the difference of two divergences with respect to the integration variable:

$$\left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla_{r'}\right)\frac{x-x'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}=\frac{x-x'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\nabla_{r'}\cdot\mathbf{J}\left(\mathbf{r}'\right)-\nabla_{r'}\cdot\left(\frac{x-x'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\mathbf{J}\left(\mathbf{r}'\right)\right) \tag{11}$$

At this point, we need to make an observation about the divergence of the current density. We saw that, due to conservation of charge:

$$\nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t} \tag{12}$$

If we're dealing with steady currents, then the amount of charge within the volume doesn't change, and

$$\nabla\cdot\mathbf{J}=0 \tag{13}$$

Making this assumption, the volume integral now becomes

$$\int_V\left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla_{r'}\right)\frac{x-x'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}'=-\int_V\nabla_{r'}\cdot\left(\frac{x-x'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\mathbf{J}\left(\mathbf{r}'\right)\right)d^3\mathbf{r}' \tag{14}$$

Now finally we can convert to a surface integral and use the fact that $\mathbf{J}=0$ outside V. Thus

$$\int_V\left(\mathbf{J}\left(\mathbf{r}'\right)\cdot\nabla_{r'}\right)\frac{x-x'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}'=0 \tag{15}$$

for the special case of steady currents. In the more general case, 13 won't be true and we have to use 12 instead, which means the first term in the integral won't be zero in general. But more on that later.

Finally, we can look at the last term in 6 and use its delta function equivalent to get

$$\nabla_r\times\mathbf{B}\left(\mathbf{r}\right)=\frac{\mu_0}{4\pi}\int_V\mathbf{J}\left(\mathbf{r}'\right)\left(\nabla\cdot\frac{\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\right)d^3\mathbf{r}' \tag{16}$$

$$=\mu_0\int_V\mathbf{J}\left(\mathbf{r}'\right)\delta_3\left(\mathbf{r}-\mathbf{r}'\right)d^3\mathbf{r}' \tag{17}$$

$$=\mu_0\mathbf{J}\left(\mathbf{r}\right) \tag{18}$$

This is Ampère's law in differential form, which relates a steady current density to the curl of the magnetic field it produces. In integral form, using Stokes's theorem, we get

$$\oint \mathbf{B}\cdot d\mathbf{l}=\mu_0\int \mathbf{J}\cdot d\mathbf{a} \tag{19}$$

That is, the integral of **B** around a closed loop is equal to the total current passing through that loop (times μ_0).

We can use this form of Ampère's law to work out the field configurations in some simple cases in much the same way we used Gauss's law to work out **E** for simple charge distributions.

As an example, suppose we have a cylindrical wire of radius a carrying a current I. If the current is uniformly distributed over the surface of the wire, what is the magnetic field inside and outside the wire?

Inside the wire we can choose a circular loop of radius $r<a$ centred on the axis of the wire. From symmetry, $\mathbf{B}\cdot d\mathbf{l}$ is a constant all around this loop, and since the enclosed current is zero (it's all on the surface of the wire), we get

$$2\pi rB = 0 \tag{20}$$

$$B = 0 \tag{21}$$

Thus there is no field inside the wire.

Outside the wire, we can again choose a circular loop of radius $r>a$. This time, the enclosed current is I so we get

$$2\pi rB = \mu_0 I \tag{22}$$

$$B = \frac{\mu_0 I}{2\pi r} \tag{23}$$

so the field decreases proportionally to $1/r$, as we saw in an earlier calculation using the Biot-Savart law.

Now suppose the current is distributed within the wire such that $J=kr$ for some constant k. First, we find k. Since the total current is I, we must have

$$I = 2\pi\int_0^a kr^2 dr \tag{24}$$

$$= \frac{2\pi ka^3}{3} \tag{25}$$

$$k = \frac{3I}{2\pi a^3} \tag{26}$$

Inside the wire, at a radius s the enclosed current is

$$I(s) = 2\pi\frac{3I}{2\pi a^3}\int_0^s r^2 dr \tag{27}$$

$$= \frac{Is^3}{a^3} \tag{28}$$

Thus the field is

$$2\pi sB(s) = \mu_0 I(s) \tag{29}$$

$$B(s) = \frac{\mu_0 Is^2}{2\pi a^3} \tag{30}$$

Outside the wire, B is the same as before since all the current is enclosed within the loop.

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