MAGNETIC VECTOR POTENTIAL

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.22.

The Biot-Savart law gives the magnetic field $\mathbf{B}$ in terms of the currents in a volume:

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{0.1}$$

By straightforward calculation, we can show that

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}=-\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{0.2}$$

Plugging this into the integral, we get

$$\mathbf{B}(\mathbf{r})=-\frac{\mu_0}{4\pi}\int_V\mathbf{J}\left(\mathbf{r}'\right)\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{0.3}$$

The gradient operator operates on the unprimed coordinates only, so we can rewrite the integrand as

$$-\mathbf{J}\left(\mathbf{r}'\right)\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\nabla\times\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{0.4}$$

To see that this works, consider the general equation

$$-\mathbf{J}\times\nabla\Phi=\nabla\times(\mathbf{J}\Phi) \tag{0.5}$$

where on both sides, the ∇ operator operates only on Φ and not on $\mathbf{J}$. Looking at each side separately and isolating the x component, we get

$$\begin{aligned}
[-\mathbf{J}\times\nabla\Phi]_x &= -J_y\partial_z\Phi+J_z\partial_y\Phi && (0.6)\\
[\nabla\times(\mathbf{J}\Phi)]_x &= \partial_y(J_z\Phi)-\partial_z(J_y\Phi) && (0.7)\\
&= J_z\partial_y\Phi-J_y\partial_z\Phi && (0.8)
\end{aligned}$$

where in the last line we can pull the components of $\mathbf{J}$ outside the derivatives since it depends only on the primed coordinates. The same derivation obviously works for the y and z components as well. Therefore we can rewrite the Biot-Savart law as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{0.9}$$

The vector quantity $\mathbf{A}(\mathbf{r})$ is defined by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{0.10}$$

and is known as the *magnetic vector potential*. This is a magnetic analog of the electrostatic condition $\mathbf{E} = -\nabla\Phi$, as we can write

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{0.11}$$

Being able to write $\mathbf{B}$ as a curl makes its divergence zero automatically.

However, we've defined $\mathbf{A}$ only by specifying what its curl is, and since we need both the curl and the divergence to determine a vector field uniquely, $\mathbf{A}$ is not uniquely specified by its derivation. Since all we require is that $\mathbf{B} = \nabla\times\mathbf{A}$, we can add any vector field to $\mathbf{A}$ that has a zero curl. Since the curl of any gradient is zero, we can write the most general form of the vector potential as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' + \nabla\lambda(\mathbf{r}) \tag{0.12}$$

where λ is any scalar function of position. Transforming the vector potential in this way is known as a *gauge transformation*. A common choice is to choose λ so that $\nabla\cdot\mathbf{A} = 0$, which can be done by requiring

$$\nabla^2\lambda = -\frac{\mu_0}{4\pi}\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{0.13}$$

The quantity on the RHS is a scalar function of $\mathbf{r}$, so this is an instance of Poisson's equation. For steady currents, we can actually work out the RHS. First, note that

$$\nabla \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) = \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \tag{0.14}$$

$$= -\mathbf{J}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \tag{0.15}$$

where in the second line, we are now taking the derivative with respect to the primed coordinates. We now get

$$\nabla \cdot \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' = -\int_V \mathbf{J}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) d^3\mathbf{r}' \tag{0.16}$$

$$= - \left. \frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \right|_\infty + \int_V \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \tag{0.17}$$

$$= 0 + 0 \tag{0.18}$$

In the second line, we've integrated by parts. The integrated term is evaluated at infinity and is zero assuming that all currents are contained within a finite volume and the second term is zero if currents are steady, since $\nabla' \cdot \mathbf{J}(\mathbf{r}') = 0$. Thus $\nabla^2 \lambda = 0$ everywhere. This is an instance of Laplace's equation, but since it applies over all space, if we require λ to be finite at infinity, the only solution is $\lambda =$ constant, which in turn implies $\nabla \lambda = 0$ so we can in fact just write

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \tag{0.19}$$

With this choice of gauge, we can get another relation for $\mathbf{A}$ by using the vector identity

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{0.20}$$

Applying $\nabla \cdot \mathbf{A} = 0$ and quoting Ampère's law, we get

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{0.21}$$

This is another instance of Poisson's equation, this time with a separate equation for each of the three components.

Finding the vector potential involves working out similar integrals to those for finding $\mathbf{B}$ from the Biot-Savart law. As an example, suppose we have a wire segment extending from z_1 to z_2 on the z axis and carrying a steady current I. The form of 0.19 for a linear current is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|} \tag{0.22}$$

We have

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{x^2+y^2+(z-z')^2} \tag{0.23}$$

Doing the integral we get

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{dz'}{\sqrt{x^2+y^2+(z-z')^2}} \tag{0.24}$$

$$= \frac{\mu_0 I}{4\pi}\ln\left[\frac{\sqrt{x^2+y^2+(z-z_2)^2}+z_2-z}{\sqrt{x^2+y^2+(z-z_1)^2}+z_1-z}\right]\hat{\mathbf{z}} \tag{0.25}$$

To check that this is correct, we can calculate

(0.26)

$$\mathbf{B} = \nabla\times\mathbf{A}$$

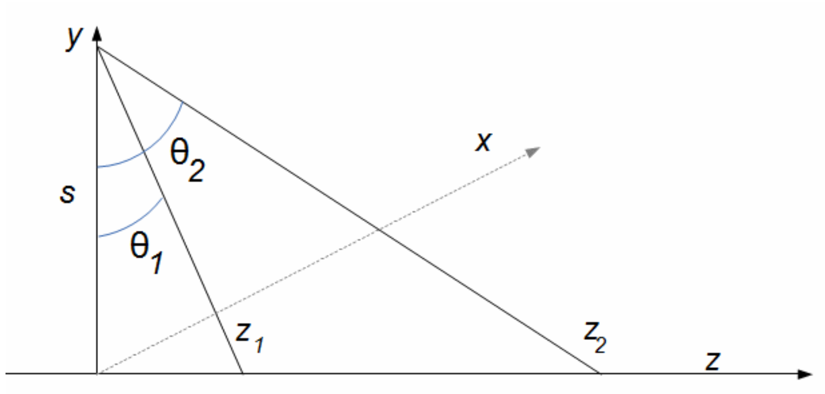
(0.27)

$$= \left(\frac{y}{r_2^2\left(\frac{z_2-z}{r_2}+1\right)} - \frac{y}{r_1^2\left(\frac{z_1-z}{r_1}+1\right)}\right)\hat{\mathbf{x}} + \left(\frac{x}{r_1^2\left(\frac{z_1-z}{r_1}+1\right)} - \frac{x}{r_2^2\left(\frac{z_2-z}{r_2}+1\right)}\right)\hat{\mathbf{y}}$$

where

$$r_i \equiv \sqrt{x^2+y^2+(z-z_i)^2} \tag{0.28}$$

To compare this with eqn 5.35 in Griffiths, we need to consider a particular field point $\mathbf{r}$, so suppose we look at $\mathbf{r} = [0,s,0]$. Then we can define the angles θ_i to be the angle between a line from $\mathbf{r}$ to z_i and the xy plane (these are the same angles Griffiths uses in his example 5.5).



From the diagram, we see that

$$\frac{z_i - z}{r_i} = \sin\theta_i \tag{0.29}$$

$$\frac{s}{r_i} = \cos\theta_i \tag{0.30}$$

For the point $\mathbf{r}$ we get

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\left[\frac{s}{r_2^2(\sin\theta_2+1)} - \frac{s}{r_1^2(\sin\theta_1+1)}\right]\hat{\mathbf{x}} \tag{0.31}$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{\cos^2\theta_2}{s(\sin\theta_2+1)} - \frac{\cos^2\theta_1}{s(\sin\theta_1+1)}\right]\hat{\mathbf{x}} \tag{0.32}$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{1-\sin^2\theta_2}{s(\sin\theta_2+1)} - \frac{1-\sin^2\theta_1}{s(\sin\theta_1+1)}\right]\hat{\mathbf{x}} \tag{0.33}$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{(1-\sin\theta_2)(1+\sin\theta_2)}{s(\sin\theta_2+1)} - \frac{(1-\sin\theta_1)(1+\sin\theta_1)}{s(\sin\theta_1+1)}\right]\hat{\mathbf{x}} \tag{0.34}$$

$$= \frac{\mu_0 I}{4\pi s}(\sin\theta_1 - \sin\theta_2)\hat{\mathbf{x}} \tag{0.35}$$

(The sign is opposite to Griffiths' eqn 5.35 because of the orientation of the axes. My x axis points into the page, so the magnetic field points out of the page, in agreement with Griffiths.)

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