

## MAGNETIC VECTOR POTENTIAL

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.22.

The Biot-Savart law gives the magnetic field  $\mathbf{B}$  in terms of the currents in a volume:

$$(1) \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}'$$

By straightforward calculation, we can show that

$$(2) \quad \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = -\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

Plugging this into the integral, we get

$$(3) \quad \mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

The gradient operator operates on the unprimed coordinates only, so we can rewrite the integrand as

$$(4) \quad -\mathbf{J}(\mathbf{r}') \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \nabla \times \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

To see that this works, consider the general equation

$$(5) \quad -\mathbf{J} \times \nabla \Phi = \nabla \times (\mathbf{J}\Phi)$$

where on both sides, the  $\nabla$  operator operates only on  $\Phi$  and not on  $\mathbf{J}$ . Looking at each side separately and isolating the  $x$  component, we get

$$(6) \quad [-\mathbf{J} \times \nabla \Phi]_x = -J_y \partial_z \Phi + J_z \partial_y \Phi$$

$$(7) \quad [\nabla \times (\mathbf{J}\Phi)]_x = \partial_y (J_z \Phi) - \partial_z (J_y \Phi)$$

$$(8) \quad = J_z \partial_y \Phi - J_y \partial_z \Phi$$

where in the last line we can pull the components of  $\mathbf{J}$  outside the derivatives since it depends only on the primed coordinates. The same derivation obviously works for the  $y$  and  $z$  components as well. Therefore we can rewrite the Biot-Savart law as

$$(9) \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

The vector quantity  $\mathbf{A}(\mathbf{r})$  is defined by

$$(10) \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

and is known as the *magnetic vector potential*. This is a magnetic analog of the electrostatic condition  $\mathbf{E} = -\nabla\Phi$ , as we can write

$$(11) \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Being able to write  $\mathbf{B}$  as a curl makes its divergence zero automatically.

However, we've defined  $\mathbf{A}$  only by specifying what its curl is, and since we need both the curl and the divergence to determine a vector field uniquely,  $\mathbf{A}$  is not uniquely specified by its derivation. Since all we require is that  $\mathbf{B} = \nabla \times \mathbf{A}$ , we can add any vector field to  $\mathbf{A}$  that has a zero curl. Since the curl of any gradient is zero, we can write the most general form of the vector potential as

$$(12) \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' + \nabla\lambda(\mathbf{r})$$

where  $\lambda$  is any scalar function of position. Transforming the vector potential in this way is known as a *gauge transformation*. A common choice is to choose  $\lambda$  so that  $\nabla \cdot \mathbf{A} = 0$ , which can be done by requiring

$$(13) \quad \nabla^2\lambda = -\frac{\mu_0}{4\pi} \nabla \cdot \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

The quantity on the RHS is a scalar function of  $\mathbf{r}$ , so this is an instance of Poisson's equation. For steady currents, we can actually work out the RHS. First, note that

$$(14) \quad \nabla \cdot \left( \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) = \mathbf{J}(\mathbf{r}') \cdot \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

$$(15) \quad = -\mathbf{J}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

where in the second line, we are now taking the derivative with respect to the primed coordinates. We now get

$$(16) \quad \nabla \cdot \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = - \int_V \mathbf{J}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3\mathbf{r}'$$

$$(17) \quad = - \left. \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} \right|_{\infty} + \int_V \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

$$(18) \quad = 0 + 0$$

In the second line, we've integrated by parts. The integrated term is evaluated at infinity and is zero assuming that all currents are contained within a finite volume and the second term is zero if currents are steady, since  $\nabla' \cdot \mathbf{J}(\mathbf{r}') = 0$ . Thus  $\nabla^2 \lambda = 0$  everywhere. This is an instance of Laplace's equation, but since it applies over all space, if we require  $\lambda$  to be finite at infinity, the only solution is  $\lambda = \text{constant}$ , which in turn implies  $\nabla \lambda = 0$  so we can in fact just write

$$(19) \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

With this choice of gauge, we can get another relation for  $\mathbf{A}$  by using the vector identity

$$(20) \quad \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Applying  $\nabla \cdot \mathbf{A} = 0$  and quoting Ampère's law, we get

$$(21) \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

This is another instance of Poisson's equation, this time with a separate equation for each of the three components.

Finding the vector potential involves working out similar integrals to those for finding  $\mathbf{B}$  from the Biot-Savart law. As an example, suppose we have a wire segment extending from  $z_1$  to  $z_2$  on the  $z$  axis and carrying a steady current  $I$ . The form of 19 for a linear current is

$$(22) \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|}$$

We have

$$(23) \quad |\mathbf{r} - \mathbf{r}'| = \sqrt{x^2 + y^2 + (z - z')^2}$$

Doing the integral we get

$$(24) \quad \mathbf{A} = \frac{\mu_0 I}{4\pi} \int_{z_1}^{z_2} \frac{dz'}{\sqrt{x^2 + y^2 + (z - z')^2}}$$

$$(25) \quad = \frac{\mu_0 I}{4\pi} \ln \left[ \frac{\sqrt{x^2 + y^2 + (z - z_2)^2} + z_2 - z}{\sqrt{x^2 + y^2 + (z - z_1)^2} + z_1 - z} \right] \hat{\mathbf{z}}$$

To check that this is correct, we can calculate

$$(26)$$

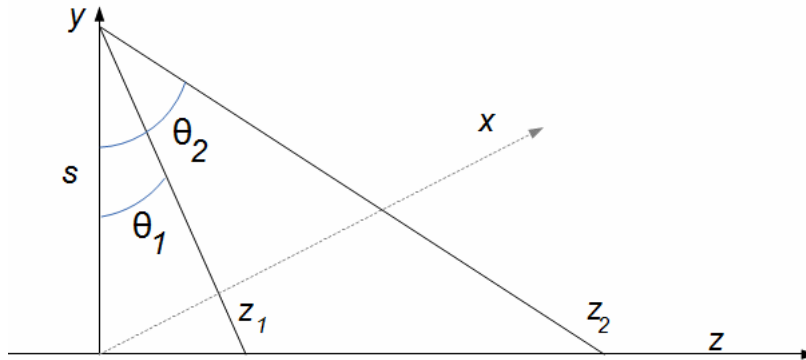
$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$(27) \quad = \left( \frac{y}{r_2^2 \left( \frac{z_2 - z}{r_2} + 1 \right)} - \frac{y}{r_1^2 \left( \frac{z_1 - z}{r_1} + 1 \right)} \right) \hat{\mathbf{x}} + \left( \frac{x}{r_1^2 \left( \frac{z_1 - z}{r_1} + 1 \right)} - \frac{x}{r_2^2 \left( \frac{z_2 - z}{r_2} + 1 \right)} \right) \hat{\mathbf{y}}$$

where

$$(28) \quad r_i \equiv \sqrt{x^2 + y^2 + (z - z_i)^2}$$

To compare this with eqn 5.35 in Griffiths, we need to consider a particular field point  $\mathbf{r}$ , so suppose we look at  $\mathbf{r} = [0, s, 0]$ . Then we can define the angles  $\theta_i$  to be the angle between a line from  $\mathbf{r}$  to  $z_i$  and the  $xy$  plane (these are the same angles Griffiths uses in his example 5.5).



From the diagram, we see that

$$(29) \quad \frac{z_i - z}{r_i} = \sin \theta_i$$

$$(30) \quad \frac{s}{r_i} = \cos \theta_i$$

For the point  $\mathbf{r}$  we get

$$(31) \quad \mathbf{B} = \frac{\mu_0 I}{4\pi} \left[ \frac{s}{r_2^2 (\sin \theta_2 + 1)} - \frac{s}{r_1^2 (\sin \theta_1 + 1)} \right] \hat{\mathbf{x}}$$

$$(32) \quad = \frac{\mu_0 I}{4\pi} \left[ \frac{\cos^2 \theta_2}{s (\sin \theta_2 + 1)} - \frac{\cos^2 \theta_1}{s (\sin \theta_1 + 1)} \right] \hat{\mathbf{x}}$$

$$(33) \quad = \frac{\mu_0 I}{4\pi} \left[ \frac{1 - \sin^2 \theta_2}{s (\sin \theta_2 + 1)} - \frac{1 - \sin^2 \theta_1}{s (\sin \theta_1 + 1)} \right] \hat{\mathbf{x}}$$

$$(34) \quad = \frac{\mu_0 I}{4\pi} \left[ \frac{(1 - \sin \theta_2)(1 + \sin \theta_2)}{s (\sin \theta_2 + 1)} - \frac{(1 - \sin \theta_1)(1 + \sin \theta_1)}{s (\sin \theta_1 + 1)} \right] \hat{\mathbf{x}}$$

$$(35) \quad = \frac{\mu_0 I}{4\pi s} (\sin \theta_1 - \sin \theta_2) \hat{\mathbf{x}}$$

(The sign is opposite to Griffiths' eqn 5.35 because of the orientation of the axes. My  $x$  axis points into the page, so the magnetic field points out of the page, in agreement with Griffiths.)

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