

MAGNETIC VECTOR POTENTIAL AND CURRENT

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.23.

The relation between magnetic vector potential \mathbf{A} and current density \mathbf{J} is given by

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad (1)$$

This equation makes it fairly easy to find the current density produced by a given vector potential, but we do need to be careful if we're using a coordinate system other than the rectangular one.

In rectangular coordinates, this equation is essentially 3 Poisson equations, one for each component. Because the rectangular unit vectors are constant over all space, the Laplacian operator doesn't affect them. When we use a different coordinate system, such as cylindrical or spherical, the unit vectors *do* vary from place to place so the Laplacian does have an effect on them, and we can't just separate \mathbf{A} into its components and apply the equation to each one.

There are two ways we could approach this problem: one is to convert \mathbf{A} to rectangular coordinates, and the other is to use the vector identity

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \quad (2)$$

since the two vector derivatives on the RHS are well defined for all coordinate systems.

As a simple example of this, suppose we have a vector potential in cylindrical coordinates:

$$\mathbf{A} = k \hat{\phi} \quad (3)$$

where k is a constant. The formulas for the divergence, gradient and curl are:

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad (4)$$

$$\nabla \times \mathbf{A} = \left[\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \hat{\mathbf{r}} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right] \hat{\phi} + \frac{1}{r} \left[\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right] \hat{\mathbf{z}} \quad (5)$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \quad (6)$$

Applying these formulas we get

$$\nabla \cdot \mathbf{A} = 0 \quad (7)$$

$$\nabla \times \mathbf{A} = \frac{k}{r} \hat{\mathbf{z}} \quad (8)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \frac{k}{r^2} \hat{\phi} \quad (9)$$

Therefore

$$\mathbf{J} = \frac{k}{\mu_0 r^2} \hat{\phi} \quad (10)$$

This is a current that rotates around the z axis and decreases with distance from the axis. We'd obviously need to avoid $r = 0$ so the current remains finite.