MAGNETIC VECTOR POTENTIAL AND CURRENT

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.23.

The relation between magnetic vector potential $\mathbf{A}$ and current density $\mathbf{J}$ is given by

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J} \tag{1}$$

This equation makes it fairly easy to find the current density produced by a given vector potential, but we do need to be careful if we're using a coordinate system other than the rectangular one.

In rectangular coordinates, this equation is essentially 3 Poisson equations, one for each component. Because the rectangular unit vectors are constant over all space, the Laplacian operator doesn't affect them. When we use a different coordinate system, such as cylindrical or spherical, the unit vectors *do* vary from place to place so the Laplacian does have an effect on them, and we can't just separate $\mathbf{A}$ into its components and apply the equation to each one.

There are two ways we could approach this problem: one is to convert $\mathbf{A}$ to rectangular coordinates, and the other is to use the vector identity

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) \tag{2}$$

since the two vector derivatives on the RHS are well defined for all coordinate systems.

As a simple example of this, suppose we have a vector potential in cylindrical coordinates:

$$\mathbf{A} = k\hat{\phi} \tag{3}$$

where k is a constant. The formulas for the divergence, gradient and curl are:

$$\nabla\cdot\mathbf{A}=\frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right)+\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z} \tag{4}$$

$$\nabla\times\mathbf{A}=\left[\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right]\hat{\mathbf{r}}+\left[\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right]\hat{\phi}+\frac{1}{r}\left[\frac{\partial\left(rA_\phi\right)}{\partial r}-\frac{\partial A_r}{\partial\phi}\right]\hat{\mathbf{z}} \tag{5}$$

$$\nabla f=\frac{\partial f}{\partial r}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial f}{\partial\phi}\hat{\phi}+\frac{\partial f}{\partial z}\hat{\mathbf{z}} \tag{6}$$

Applying these formulas we get

$$\nabla\cdot\mathbf{A} = 0 \tag{7}$$

$$\nabla\times\mathbf{A} = \frac{k}{r}\hat{\mathbf{z}} \tag{8}$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right) = \frac{k}{r^2}\hat{\phi} \tag{9}$$

Therefore

$$\mathbf{J}=\frac{k}{\mu_0 r^2}\hat{\phi} \tag{10}$$

This is a current that rotates around the z axis and decreases with distance from the axis. We'd obviously need to avoid $r=0$ so the current remains finite.