MAGNETIC VECTOR POTENTIAL OF AN INFINITE WIRE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.25.

The magnetic vector potential $\mathbf{A}$ can be evaluated from

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{1}$$

provided that all currents are contained within a finite volume. If the currents extend to infinity we have to use a different method.

One example is that of an infinite wire carrying a steady current I. We can work out the potential by applying Stokes's theorem. From the definition of $\mathbf{A}$, we know that $\nabla\times\mathbf{A}=\mathbf{B}$, so if we define a closed loop then we have

$$\oint\mathbf{A}\cdot d\mathbf{l}=\int_A\mathbf{B}\cdot d\mathbf{a} \tag{2}$$

where the integral on the LHS is a line integral around the loop and the integral on the right is over the area A enclosed by the loop. For a flat loop, the direction of integration on the LHS is such that the right hand rule gives a vector pointing in the same direction as $d\mathbf{a}$ on the RHS.

For an infinite wire along the z axis, the magnetic field, using cylindrical coordinates, is

$$\mathbf{B}=\frac{I\mu_0}{2\pi r}\hat{\phi} \tag{3}$$

where r is the distance from the axis of the wire.

We choose a rectangular loop with one edge a distance a from the wire and the opposite edge a distance b from the axis (with length of the parallel sides taken as 1), we can integrate $\mathbf{B}$ over this area:

$$\int_A \mathbf{B}\cdot d\mathbf{a} = \int_a^b \frac{I\mu_0}{2\pi r}dr \tag{4}$$

$$= \frac{I\mu_0}{2\pi}(\ln b - \ln a) \tag{5}$$

By symmetry, **A** has the same magnitude for all points at a given distance r from the axis, and since the 2 edges perpendicular to the wire are traversed in opposite directions, their contributions to the line integral cancel, and we get

$$\oint \mathbf{A}\cdot d\mathbf{l} = A(a) - A(b) \tag{6}$$

Thus it seems a good candidate is

$$\mathbf{A} = -\frac{I\mu_0}{2\pi}\ln r\hat{\mathbf{z}} \tag{7}$$

We can check its div and curl, and we find (using standard formulas for div and curl in cylindrical coordinates):

$$\nabla\cdot\mathbf{A} = 0 \tag{8}$$

$$\nabla\times\mathbf{A} = \frac{I\mu_0}{2\pi r}\hat{\phi} \tag{9}$$

Now suppose we look inside the wire, taking the wire's radius to be R. At a distance r from the axis, only current within that radius contributes to the field, so we get from Ampère's law using a circular loop within the wire

$$2\pi r B = \mu_0\frac{r^2}{R^2}I \tag{10}$$

$$\mathbf{B} = \frac{\mu_0 r I}{2\pi R^2}\hat{\phi} \tag{11}$$

Following the same procedure as above, only this time choosing a rectangular loop entirely within the wire, we get

$$A(a) - A(b) = \frac{\mu_0 I}{2\pi R^2}\int_a^b r\,dr \tag{12}$$

$$= \frac{\mu_0 I}{4\pi R^2}\left(b^2 - a^2\right) \tag{13}$$

Thus it seems a candidate is

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi R^2} r^2 \hat{\mathbf{z}} \tag{14}$$

Again, we get $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$, so it looks like we're done.

However, there is one slight snag here: $\mathbf{A}$ isn't continuous at $r = R$. Although in practice this might not make much difference since it is only the derivatives of $\mathbf{A}$ that have physical meaning (and we've seen that both the div and curl are correct), it's nice to tidy up the solution.

The key is that when we did the line integrals above, we ended up only with expressions for the *difference* between $\mathbf{A}$ at two distances from the wire. The potential itself could have a non-zero constant of integration which cancels out in the difference (and also wouldn't contribute to the div or curl). Thus we'd like to choose a couple of constants so that

$$\frac{I\mu_0}{2\pi}(\ln R + C_1) = \frac{I\mu_0}{4\pi R^2}\left(R^2 + C_2\right) \tag{15}$$

The simplest choice is to make both sides zero, so

$$C_1 = -\ln R \tag{16}$$

$$C_2 = -R^2 \tag{17}$$

Our final solution is then

$$\mathbf{A} = \begin{cases} -\frac{I\mu_0}{2\pi} \ln \frac{r}{R} & r \geq R \\ -\frac{\mu_0 I}{4\pi R^2}\left(r^2 - R^2\right) & r \leq R \end{cases} \tag{18}$$

Pingbacks

Pingback: Magnetic vector potential: sheet of current