

## MAGNETIC VECTOR POTENTIAL: DIV, CURL AND LAPLACIAN

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.27.

We've seen how the magnetic vector potential is derived from the fact that the magnetic field can be expressed as the curl of a vector field. The fact that  $\nabla \times \mathbf{A}$  gives the Biot-Savart equation for the magnetic field can be obtained by just reversing the derivation of  $\mathbf{A}$ .

We can check a couple of other derivatives from the definition of  $\mathbf{A}$ , which is

$$(1) \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

First, the divergence  $\nabla \cdot \mathbf{A}$ . Remember that the derivative is taken only with respect to the unprimed coordinates, so the divergence works out to

$$(2) \quad \nabla \cdot \mathbf{A} = \frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

If we look at the derivative with respect to  $x$  term, we have

$$(3) \quad \int_V J_x(\mathbf{r}') \cdot \frac{\partial}{\partial x} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3\mathbf{r}' = - \int_V J_x(\mathbf{r}') \frac{\partial}{\partial x'} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3\mathbf{r}'$$
$$(4) \quad = - J_x(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|} \Big|_A + \int_V \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial}{\partial x'} (J_x(\mathbf{r}')) d^3\mathbf{r}'$$

In the first line, we used the fact that the derivative of  $\frac{1}{|\mathbf{r} - \mathbf{r}'|}$  with respect to  $x$  is the negative of the derivative with respect to  $x'$ . We then integrated by parts. The integrated term is evaluated over the boundary surface  $A$ . We can take this surface to lie outside the region in which currents exist, so  $J_x = 0$  there. We will get similar results for the  $y$  and  $z$  terms in the original integral, so if we combine them, we get

$$(5) \quad \int_V \mathbf{J}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}' = \int_V \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{J}(\mathbf{r}') d^3 \mathbf{r}'$$

For steady currents,  $\nabla' \cdot \mathbf{J}(\mathbf{r}') = 0$ , so in the case of localized, steady currents, we have  $\nabla \cdot \mathbf{A} = 0$ , which is what we assumed in our derivation of  $\mathbf{A}$ , so this is consistent.

We also had a version of Poisson's equation for the potential:

$$(6) \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Taking the  $x$  component, we have

$$(7) \quad \nabla^2 A_x(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V J_x(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'$$

$$(8) \quad = -\mu_0 \int_V J_x(\mathbf{r}') \delta^3(|\mathbf{r} - \mathbf{r}'|) d^3 \mathbf{r}'$$

$$(9) \quad = -\mu_0 J_x(\mathbf{r})$$

where we've used a result for the delta function.

Similar relations for  $y$  and  $z$  verify Poisson's equation.