

MAGNETIC VECTOR POTENTIAL: DIV, CURL AND LAPLACIAN

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.27.

We've seen how the magnetic vector potential is derived from the fact that the magnetic field can be expressed as the curl of a vector field. The fact that $\nabla \times \mathbf{A}$ gives the Biot-Savart equation for the magnetic field can be obtained by just reversing the derivation of \mathbf{A} .

We can check a couple of other derivatives from the definition of \mathbf{A} , which is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (1)$$

First, the divergence $\nabla \cdot \mathbf{A}$. Remember that the derivative is taken only with respect to the unprimed coordinates, so the divergence works out to

$$\nabla \cdot \mathbf{A} = \frac{\mu_0}{4\pi} \int_V \mathbf{J}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (2)$$

If we look at the derivative with respect to x term, we have

$$\begin{aligned} \int_V J_x(\mathbf{r}') \cdot \frac{\partial}{\partial x} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3\mathbf{r}' &= - \int_V J_x(\mathbf{r}') \frac{\partial}{\partial x'} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3\mathbf{r}' \quad (3) \\ &= - J_x(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|} \Big|_A + \int_V \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial}{\partial x'} (J_x(\mathbf{r}')) d^3\mathbf{r}' \quad (4) \end{aligned}$$

In the first line, we used the fact that the derivative of $\frac{1}{|\mathbf{r} - \mathbf{r}'|}$ with respect to x is the negative of the derivative with respect to x' . We then integrated by parts. The integrated term is evaluated over the boundary surface A . We can take this surface to lie outside the region in which currents exist, so $J_x = 0$ there. We will get similar results for the y and z terms in the original integral, so if we combine them, we get

$$\int_V \mathbf{J}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = \int_V \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' \quad (5)$$

For steady currents, $\nabla' \cdot \mathbf{J}(\mathbf{r}') = 0$, so in the case of localized, steady currents, we have $\nabla \cdot \mathbf{A} = 0$, which is what we assumed in our derivation of \mathbf{A} , so this is consistent.

We also had a version of Poisson's equation for the potential:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad (6)$$

Taking the x component, we have

$$\nabla^2 A_x(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V J_x(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}' \quad (7)$$

$$= -\mu_0 \int_V J_x(\mathbf{r}') \delta^3(|\mathbf{r} - \mathbf{r}'|) d^3 \mathbf{r}' \quad (8)$$

$$= -\mu_0 J_x(\mathbf{r}) \quad (9)$$

where we've used a result for the delta function.

Similar relations for y and z verify Poisson's equation.