MAGNETIC VECTOR POTENTIAL: DIV, CURL AND LAPLACIAN

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.27.

We've seen how the magnetic vector potential is derived from the fact that the magnetic field can be expressed as the curl of a vector field. The fact that $\nabla\times\mathbf{A}$ gives the Biot-Savart equation for the magnetic field can be obtained by just reversing the derivation of $\mathbf{A}$.

We can check a couple of other derivatives from the definition of $\mathbf{A}$, which is

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{1}$$

First, the divergence $\nabla\cdot\mathbf{A}$. Remember that the derivative is taken only with respect to the unprimed coordinates, so the divergence works out to

$$\nabla\cdot\mathbf{A}=\frac{\mu_0}{4\pi}\int_V\mathbf{J}(\mathbf{r}')\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{2}$$

If we look at the derivative with respect to x term, we have

$$\int_V J_x(\mathbf{r}')\cdot\frac{\partial}{\partial x}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3\mathbf{r}'=-\int_V J_x(\mathbf{r}')\frac{\partial}{\partial x'}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3\mathbf{r}' \tag{3}$$

$$=-\left.J_x(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right|_A+\int_V\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial x'}\left(J_x(\mathbf{r}')\right)d^3\mathbf{r}' \tag{4}$$

In the first line, we used the fact that the derivative of $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ with respect to x is the negative of the derivative with respect to x'. We then integrated by parts. The integrated term is evaluated over the boundary surface A. We can take this surface to lie outside the region in which currents exist, so $J_x=0$ there. We will get similar results for the y and z terms in the original integral, so if we combine them, we get

$$\int_V\mathbf{J}(\mathbf{r}')\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'=\int_V\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbf{J}(\mathbf{r}')\,d^3\mathbf{r}' \tag{5}$$

For steady currents, $\nabla' \cdot \mathbf{J}(\mathbf{r}') = 0$, so in the case of localized, steady currents, we have $\nabla \cdot \mathbf{A} = 0$, which is what we assumed in our derivation of $\mathbf{A}$, so this is consistent.

We also had a version of Poisson's equation for the potential:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{6}$$

Taking the x component, we have

$$\begin{aligned}
\nabla^2 A_x(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V J_x\left(\mathbf{r}'\right)\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' && (7)\\
&= -\mu_0\int_V J_x\left(\mathbf{r}'\right)\delta^3\left(\left|\mathbf{r}-\mathbf{r}'\right|\right)d^3\mathbf{r}' && (8)\\
&= -\mu_0 J_x(\mathbf{r}) && (9)
\end{aligned}$$

where we've used a result for the delta function.

Similar relations for y and z verify Poisson's equation.