

## MAGNETIC FIELD OF ROTATING SPHERE OF CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.29.

Since any moving charge generates a magnetic field, one way of producing a novel current is to take a uniform sphere of charge and set it spinning on its axis. To work out the field produced by such a sphere, we can start with the field generated by a spinning spherical shell of charge. The derivation of this field is surprisingly tricky, and is given by Griffiths as example 5.11. The results for the vector potential are

$$(0.1) \quad \mathbf{A}(r, \theta, \phi) = \begin{cases} \frac{1}{3}\mu_0 R \omega \sigma r \sin \theta \hat{\phi} & r \leq R \\ \frac{\mu_0 R^4 \omega \sigma}{3} \frac{\sin \theta}{r^2} \hat{\phi} & r \geq R \end{cases}$$

Here,  $R$  is the radius of the shell,  $\sigma$  is the surface charge density and  $\omega$  is the angular velocity, where the sphere's axis is taken to be the  $z$  axis.

From this we can calculate the magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$  using the standard formula for the curl in spherical coordinates:

$$(0.2) \quad \mathbf{B} = \begin{cases} \frac{2\mu_0 R \omega \sigma}{3} (\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta}) & r \leq R \\ \frac{\mu_0 R^4 \omega \sigma}{3} \frac{1}{r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) & r \geq R \end{cases}$$

Notice that although  $\mathbf{A}$  is continuous at  $r = R$ ,  $\mathbf{B}$  is not. We'll look at this in more detail in a future post. It's also worth noting that since  $\hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta}$ , the field inside the shell points along the rotation axis and is uniform.

To find the field due to a solid spinning sphere of charge with charge density  $\rho$ , we can integrate a series of spherical shells. In doing this, we need to be very careful in interpreting the various symbols for the radius.

For a thin shell of thickness  $dr$ , the charge per unit area on this shell is  $\rho dr$ , so this will replace  $\sigma$  in the equations above. In these equations,  $R$  is the radius of the shell, and  $r$  is the observation radius. We need the overall radius of the solid sphere, which we'll define as  $R_0$ . Thus the integration variable will be  $R$ , since it is the radius of the shells that we need to vary.

Then for a value of  $r$  inside the sphere, we will get a contribution to the field from those shells with a radius  $R < r$  by using the second equation

above, and for  $R > r$  by using the first equation. Note that we do *not* integrate over either  $r$  or  $\theta$ , since these two coordinates define the observation point. We get for the field due to shells interior to the observation radius:

$$(0.3) \quad \mathbf{B}_i = \frac{\mu_0 \omega \rho}{3} \frac{1}{r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \int_0^r R^4 dR$$

$$(0.4) \quad = \frac{\mu_0 \omega \rho}{3} \frac{r^2}{5} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta})$$

From shells outside the observation radius:

$$(0.5) \quad \mathbf{B}_o = \frac{2\mu_0 \omega \rho}{3} (\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta}) \int_r^{R_0} R dR$$

$$(0.6) \quad = \frac{\mu_0 \omega \rho}{3} (R_0^2 - r^2) (\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta})$$

The total field is the sum so

$$(0.7) \quad \mathbf{B} = \frac{\mu_0 \omega \rho}{3} \left[ \left( R_0^2 - \frac{3}{5} r^2 \right) \cos \theta \hat{\mathbf{r}} + \left( \frac{6}{5} r^2 - R_0^2 \right) \sin \theta \hat{\theta} \right]$$

If we know the total charge  $Q$  in the sphere, then  $\rho = \frac{3Q}{4\pi R_0^3}$  and

$$(0.8) \quad \mathbf{B} = \frac{\mu_0 \omega Q}{4\pi R_0} \left[ \left( 1 - \frac{3}{5} \frac{r^2}{R_0^2} \right) \cos \theta \hat{\mathbf{r}} + \left( \frac{6}{5} \frac{r^2}{R_0^2} - 1 \right) \sin \theta \hat{\theta} \right]$$

$$(0.9) \quad = \frac{\mu_0 \omega Q}{4\pi R_0} \left[ \hat{\mathbf{z}} - \frac{3}{5} \frac{r^2}{R_0^2} \cos \theta \hat{\mathbf{r}} + \frac{6}{5} \frac{r^2}{R_0^2} \sin \theta \hat{\theta} \right]$$

Although the last version mixes spherical and rectangular coordinates, it shows that there is a uniform field in the  $z$  direction with a varying field superimposed on top of it.

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