MAGNETIC FIELD OF ROTATING SPHERE OF CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.29.

Since any moving charge generates a magnetic field, one way of producing a novel current is to take a uniform sphere of charge and set it spinning on its axis. To work out the field produced by such a sphere, we can start with the field generated by a spinning spherical shell of charge. The derivation of this field is surprisingly tricky, and is given by Griffiths as example 5.11. The results for the vector potential are

$$\mathbf{A}(r,\theta,\phi)=\begin{cases}\frac{1}{3}\mu_0 R\omega\sigma r\sin\theta\hat{\phi} & r\le R\\ \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} & r\ge R\end{cases} \tag{1}$$

Here, R is the radius of the shell, σ is the surface charge density and ω is the angular velocity, where the sphere's axis is taken to be the z axis.

From this we can calculate the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ using the standard formula for the curl in spherical coordinates:

$$\mathbf{B}=\begin{cases}\frac{2\mu_0 R\omega\sigma}{3}\left(\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\theta}\right) & r\le R\\ \frac{\mu_0 R^4\omega\sigma}{3}\frac{1}{r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right) & r\ge R\end{cases} \tag{2}$$

Notice that although $\mathbf{A}$ is continuous at $r=R$, $\mathbf{B}$ is not. We'll look at this in more detail in a future post. It's also worth noting that since $\hat{\mathbf{z}}=\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\theta}$, the field inside the shell points along the rotation axis and is uniform.

To find the field due to a solid spinning sphere of charge with charge density ρ, we can integrate a series of spherical shells. In doing this, we need to be very careful in interpreting the various symbols for the radius.

For a thin shell of thickness dr, the charge per unit area on this shell is ρdr, so this will replace σ in the equations above. In these equations, R is the radius of the shell, and r is the observation radius. We need the overall radius of the solid sphere, which we'll define as R_0. Thus the integration variable will be R, since it is the radius of the shells that we need to vary.

Then for a value of r inside the sphere, we will get a contribution to the field from those shells with a radius $R<r$ by using the second equation

above, and for $R>r$ by using the first equation. Note that we do *not* integrate over either r or θ, since these two coordinates define the observation point. We get for the field due to shells interior to the observation radius:

$$\mathbf{B}_i = \frac{\mu_0\omega\rho}{3}\frac{1}{r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right)\int_0^r R^4 dR \tag{3}$$

$$= \frac{\mu_0\omega\rho}{3}\frac{r^2}{5}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right) \tag{4}$$

From shells outside the observation radius:

$$\mathbf{B}_o = \frac{2\mu_0\omega\rho}{3}\left(\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\theta}\right)\int_r^{R_0} RdR \tag{5}$$

$$= \frac{\mu_0\omega\rho}{3}\left(R_0^2-r^2\right)\left(\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\theta}\right) \tag{6}$$

The total field is the sum so

$$\mathbf{B} = \frac{\mu_0\omega\rho}{3}\left[\left(R_0^2-\frac{3}{5}r^2\right)\cos\theta\hat{\mathbf{r}}+\left(\frac{6}{5}r^2-R_0^2\right)\sin\theta\hat{\theta}\right] \tag{7}$$

If we know the total charge Q in the sphere, then $\rho=\frac{3Q}{4\pi R_0^3}$ and

$$\mathbf{B} = \frac{\mu_0\omega Q}{4\pi R_0}\left[\left(1-\frac{3}{5}\frac{r^2}{R_0^2}\right)\cos\theta\hat{\mathbf{r}}+\left(\frac{6}{5}\frac{r^2}{R_0^2}-1\right)\sin\theta\hat{\theta}\right] \tag{8}$$

$$= \frac{\mu_0\omega Q}{4\pi R_0}\left[\hat{\mathbf{z}}-\frac{3}{5}\frac{r^2}{R_0^2}\cos\theta\hat{\mathbf{r}}+\frac{6}{5}\frac{r^2}{R_0^2}\sin\theta\hat{\theta}\right] \tag{9}$$

Although the last version mixes spherical and rectangular coordinates, it shows that there is a uniform field in the z direction with a varying field superimposed on top of it.

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