DIVERGENCELESS VECTOR FIELD AS A CURL

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.30.

This is more mathematics than physics, but it relates to the magnetic vector potential so here we go. The vector potential $\mathbf{A}$ is defined so that $\mathbf{B} = \nabla\times\mathbf{A}$ and $\nabla\cdot\mathbf{B} = 0$. We'd like to prove that, in general, any divergenceless vector field $\mathbf{F}$ can be written as the curl of another vector field $\mathbf{A}$. The curl $\mathbf{F} = \nabla\times\mathbf{A}$ has three components:

$$\partial_y A_z - \partial_z A_y = F_x \tag{1}$$
$$\partial_x A_z - \partial_z A_x = -F_y \tag{2}$$
$$\partial_x A_y - \partial_y A_x = F_z \tag{3}$$

Thus in principle, we have 3 coupled PDEs to solve, but we can get a solution by starting with the assumption that $A_x = 0$ (this isn't the only solution, of course, but it does give *a* solution). Then

$$\partial_x A_z = -F_y \tag{4}$$
$$A_z = -\int F_y\left(x',y,z\right)dx' + G(y,z) \tag{5}$$
$$\partial_x A_y = F_z \tag{6}$$
$$A_y = \int F_z\left(x',y,z\right)dx' + H(y,z) \tag{7}$$

where G and H are functions of integration; since the integrals are with respect to x, the 'constants' of integration can be functions of y and z.

We can now plug these into the first component of the curl above:

$$\partial_y A_z - \partial_z A_y = -\int \partial_z F_z\left(x',y,z\right)dx' + \partial_y G(y,z) - \int \partial_y F_z\left(x',y,z\right)dx' - \partial_z H(y,z) \tag{8}$$
$$= \int\left(-\nabla\cdot\mathbf{F} + \partial_{x'}F_{x'}\right)dx' + \partial_y G(y,z) - \partial_z H(y,z) \tag{9}$$

If we now want the result at a particular point (x,y,z), we can introduce limits on the integral, and also use the requirement that $\nabla\cdot\mathbf{F}=0$:

$$F_x(x,y,z) = \int_0^x \partial_{x'} F_{x'} dx' + \partial_y G(y,z) - \partial_z H(y,z) \tag{10}$$

$$= F_x(x,y,z) - F_x(0,y,z) + \partial_y G(y,z) - \partial_z H(y,z) \tag{11}$$

$$0 = -F_x(0,y,z) + \partial_y G(y,z) - \partial_z H(y,z) \tag{12}$$

At this point we can proceed in various ways, since the functions G and H have between them only this one condition. One option is to choose

$$G(y,z) = \int_0^y F_x\left(0,y',z\right) dy' \tag{13}$$

$$H(y,z) = 0 \tag{14}$$

With this option, we get

$$A_z = \int_0^y F_x\left(0,y',z\right) dy' - \int_0^x F_y\left(x',y,z\right) dx' \tag{15}$$

$$A_y = \int_0^x F_z\left(x',y,z\right) dx' \tag{16}$$

$$A_x = 0 \tag{17}$$

We could equally as well have chosen

$$G(y,z) = 0 \tag{18}$$

$$H(y,z) = -\int_0^z F_x\left(0,y,z'\right) dz' \tag{19}$$

or some combination of the two.

Using the first option, we can check the curl.

(20)
$$(\nabla\times\mathbf{A})_x=\partial_y\left[\int_0^y F_x\left(0,y',z\right)dy'-\int_0^x F_y\left(x',y,z\right)dx'\right]-\partial_z\left[\int_0^x F_z\left(x',y,z\right)dx'\right]$$
(21)
$$=F_x(0,y,z)+\int_0^x\left(-\nabla\cdot\mathbf{F}+\partial_{x'}F_{x'}\right)dx'$$
(22)
$$=F_x(0,y,z)+F_x(x,y,z)-F_x(0,y,z)$$
(23)
$$=F_x(x,y,z)$$
(24)
$$(\nabla\times\mathbf{A})_y=-\partial_x\left[\int_0^y F_x\left(0,y',z\right)dy'-\int_0^x F_y\left(x',y,z\right)dx'\right]+\partial_z(0)$$
(25)
$$=F_y(x,y,z)$$
(26)
$$(\nabla\times\mathbf{A})_z=\partial_x\int_0^x F_z\left(x',y,z\right)dx'-\partial_y(0)$$
(27)
$$=F_z(x,y,z)$$

For the divergence

(28)
$$\nabla\cdot\mathbf{A}=\partial_x(0)+\partial_y\int_0^x F_z\left(x',y,z\right)dx'+\partial_z\left[\int_0^y F_x\left(0,y',z\right)dy'-\int_0^x F_y\left(x',y,z\right)dx'\right]$$

This isn't zero in general.
For a specific example, consider

$$\mathbf{F}=y\hat{\mathbf{x}}+z\hat{\mathbf{y}}+x\hat{\mathbf{z}} \tag{29}$$

Then

$$A_x = 0 \tag{30}$$
$$A_y = \int_0^x F_z\left(x', y, z\right) dx' \tag{31}$$
$$= \int_0^x x' dx' \tag{32}$$
$$= \frac{x^2}{2} \tag{33}$$
$$A_z = \int_0^y F_x\left(0, y', z\right) dy' - \int_0^x F_y\left(x', y, z\right) dx' \tag{34}$$
$$= \int_0^y y' dy' - \int_0^x z dx' \tag{35}$$
$$= \frac{y^2}{2} - xz \tag{36}$$

By direct calculation

$$(\nabla \times \mathbf{A})_x = \partial_y A_z - \partial_z A_y \tag{37}$$
$$= y = F_x \tag{38}$$
$$(\nabla \times \mathbf{A})_y = \partial_z A_x - \partial_x A_z \tag{39}$$
$$= z = F_y \tag{40}$$
$$(\nabla \times \mathbf{A})_z = \partial_x A_y - \partial_y A_x \tag{41}$$
$$= x = F_z \tag{42}$$