DIVERGENCELESS VECTOR FIELD AS A CURL

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.30.

This is more mathematics than physics, but it relates to the magnetic vector potential so here we go. The vector potential $\mathbf{A}$ is defined so that $\mathbf{B}=\nabla\times\mathbf{A}$ and $\nabla\cdot\mathbf{B}=0$. We'd like to prove that, in general, any divergenceless vector field $\mathbf{F}$ can be written as the curl of another vector field $\mathbf{A}$. The curl $\mathbf{F}=\nabla\times\mathbf{A}$ has three components:

$$\begin{aligned}
\partial_y A_z-\partial_z A_y &= F_x &(1)\\
\partial_x A_z-\partial_z A_x &= -F_y &(2)\\
\partial_x A_y-\partial_y A_x &= F_z &(3)
\end{aligned}$$

Thus in principle, we have 3 coupled PDEs to solve, but we can get a solution by starting with the assumption that $A_x=0$ (this isn't the only solution, of course, but it does give *a* solution). Then

$$\begin{aligned}
\partial_x A_z &= -F_y &(4)\\
A_z &= -\int F_y\left(x',y,z\right)dx'+G(y,z) &(5)\\
\partial_x A_y &= F_z &(6)\\
A_y &= \int F_z\left(x',y,z\right)dx'+H(y,z) &(7)
\end{aligned}$$

where G and H are functions of integration; since the integrals are with respect to x, the 'constants' of integration can be functions of y and z.

We can now plug these into the first component of the curl above:

$$\partial_y A_z-\partial_z A_y=-\int\partial_z F_z\left(x',y,z\right)dx'+\partial_y G(y,z)-\int\partial_y F_z\left(x',y,z\right)dx'-\partial_z H(y,z) \tag{8}$$

$$=\int\left(-\nabla\cdot\mathbf{F}+\partial_{x'}F_{x'}\right)dx'+\partial_y G(y,z)-\partial_z H(y,z) \tag{9}$$

If we now want the result at a particular point (x,y,z), we can introduce limits on the integral, and also use the requirement that $\nabla\cdot\mathbf{F}=0$:

$$F_x(x,y,z) = \int_0^x \partial_{x'} F_{x'} dx' + \partial_y G(y,z) - \partial_z H(y,z) \tag{10}$$

$$= F_x(x,y,z) - F_x(0,y,z) + \partial_y G(y,z) - \partial_z H(y,z) \tag{11}$$

$$0 = -F_x(0,y,z) + \partial_y G(y,z) - \partial_z H(y,z) \tag{12}$$

At this point we can proceed in various ways, since the functions G and H have between them only this one condition. One option is to choose

$$G(y,z) = \int_0^y F_x\left(0,y',z\right) dy' \tag{13}$$

$$H(y,z) = 0 \tag{14}$$

With this option, we get

$$A_z = \int_0^y F_x\left(0,y',z\right) dy' - \int_0^x F_y\left(x',y,z\right) dx' \tag{15}$$

$$A_y = \int_0^x F_z\left(x',y,z\right) dx' \tag{16}$$

$$A_x = 0 \tag{17}$$

We could equally as well have chosen

$$G(y,z) = 0 \tag{18}$$

$$H(y,z) = -\int_0^z F_x\left(0,y,z'\right) dz' \tag{19}$$

or some combination of the two.

Using the first option, we can check the curl.

$$(\nabla\times\mathbf{A})_x = \partial_y\left[\int_0^y F_x\left(0,y',z\right)dy' - \int_0^x F_y\left(x',y,z\right)dx'\right] - \partial_z\left[\int_0^x F_z\left(x',y,z\right)dx'\right] \tag{20}$$

$$= F_x(0,y,z) + \int_0^x \left(-\nabla\cdot\mathbf{F} + \partial_{x'}F_{x'}\right)dx' \tag{21}$$

$$= F_x(0,y,z) + F_x(x,y,z) - F_x(0,y,z) \tag{22}$$

$$= F_x(x,y,z) \tag{23}$$

$$(\nabla\times\mathbf{A})_y = -\partial_x\left[\int_0^y F_x\left(0,y',z\right)dy' - \int_0^x F_y\left(x',y,z\right)dx'\right] + \partial_z(0) \tag{24}$$

$$= F_y(x,y,z) \tag{25}$$

$$(\nabla\times\mathbf{A})_z = \partial_x \int_0^x F_z\left(x',y,z\right)dx' - \partial_y(0) \tag{26}$$

$$= F_z(x,y,z) \tag{27}$$

For the divergence

$$\nabla\cdot\mathbf{A} = \partial_x(0) + \partial_y\int_0^x F_z\left(x',y,z\right)dx' + \partial_z\left[\int_0^y F_x\left(0,y',z\right)dy' - \int_0^x F_y\left(x',y,z\right)dx'\right] \tag{28}$$

This isn't zero in general.
For a specific example, consider

$$\mathbf{F} = y\hat{\mathbf{x}} + z\hat{\mathbf{y}} + x\hat{\mathbf{z}} \tag{29}$$

Then

$$\begin{aligned}
A_x &= 0 && (30)\\
A_y &= \int_0^x F_z\left(x',y,z\right)dx' && (31)\\
&= \int_0^x x'dx' && (32)\\
&= \frac{x^2}{2} && (33)\\
A_z &= \int_0^y F_x\left(0,y',z\right)dy' - \int_0^x F_y\left(x',y,z\right)dx' && (34)\\
&= \int_0^y y'dy' - \int_0^x zdx' && (35)\\
&= \frac{y^2}{2} - xz && (36)
\end{aligned}$$

By direct calculation

$$\begin{aligned}
\left(\nabla\times\mathbf{A}\right)_x &= \partial_y A_z - \partial_z A_y && (37)\\
&= y = F_x && (38)\\
\left(\nabla\times\mathbf{A}\right)_y &= \partial_z A_x - \partial_x A_z && (39)\\
&= z = F_y && (40)\\
\left(\nabla\times\mathbf{A}\right)_z &= \partial_x A_y - \partial_y A_x && (41)\\
&= x = F_z && (42)
\end{aligned}$$