

## DIVERGENCELESS VECTOR FIELD AS A CURL

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.30.

This is more mathematics than physics, but it relates to the magnetic vector potential so here we go. The vector potential  $\mathbf{A}$  is defined so that  $\mathbf{B} = \nabla \times \mathbf{A}$  and  $\nabla \cdot \mathbf{B} = 0$ . We'd like to prove that, in general, any divergenceless vector field  $\mathbf{F}$  can be written as the curl of another vector field  $\mathbf{A}$ . The curl  $\mathbf{F} = \nabla \times \mathbf{A}$  has three components:

$$(0.1) \quad \partial_y A_z - \partial_z A_y = F_x$$

$$(0.2) \quad \partial_x A_z - \partial_z A_x = -F_y$$

$$(0.3) \quad \partial_x A_y - \partial_y A_x = F_z$$

Thus in principle, we have 3 coupled PDEs to solve, but we can get a solution by starting with the assumption that  $A_x = 0$  (this isn't the only solution, of course, but it does give a solution). Then

$$(0.4) \quad \partial_x A_z = -F_y$$

$$(0.5) \quad A_z = -\int F_y(x', y, z) dx' + G(y, z)$$

$$(0.6) \quad \partial_x A_y = F_z$$

$$(0.7) \quad A_y = \int F_z(x', y, z) dx' + H(y, z)$$

where  $G$  and  $H$  are functions of integration; since the integrals are with respect to  $x$ , the 'constants' of integration can be functions of  $y$  and  $z$ .

We can now plug these into the first component of the curl above:

$$(0.8)$$

$$\partial_y A_z - \partial_z A_y = -\int \partial_z F_z(x', y, z) dx' + \partial_y G(y, z) - \int \partial_y F_z(x', y, z) dx' - \partial_z H(y, z)$$

$$(0.9)$$

$$= \int (-\nabla \cdot \mathbf{F} + \partial_{x'} F_{x'}) dx' + \partial_y G(y, z) - \partial_z H(y, z)$$

If we now want the result at a particular point  $(x, y, z)$ , we can introduce limits on the integral, and also use the requirement that  $\nabla \cdot \mathbf{F} = 0$ :

$$(0.10) \quad F_x(x, y, z) = \int_0^x \partial_{x'} F_{x'} dx' + \partial_y G(y, z) - \partial_z H(y, z)$$

$$(0.11) \quad = F_x(x, y, z) - F_x(0, y, z) + \partial_y G(y, z) - \partial_z H(y, z)$$

$$(0.12) \quad 0 = -F_x(0, y, z) + \partial_y G(y, z) - \partial_z H(y, z)$$

At this point we can proceed in various ways, since the functions  $G$  and  $H$  have between them only this one condition. One option is to choose

$$(0.13) \quad G(y, z) = \int_0^y F_x(0, y', z) dy'$$

$$(0.14) \quad H(y, z) = 0$$

With this option, we get

$$(0.15) \quad A_z = \int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx'$$

$$(0.16) \quad A_y = \int_0^x F_z(x', y, z) dx'$$

$$(0.17) \quad A_x = 0$$

We could equally as well have chosen

$$(0.18) \quad G(y, z) = 0$$

$$(0.19) \quad H(y, z) = - \int_0^z F_x(0, y, z') dz'$$

or some combination of the two.

Using the first option, we can check the curl.

(0.20)

$$(\nabla \times \mathbf{A})_x = \partial_y \left[ \int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx' \right] - \partial_z \left[ \int_0^x F_z(x', y, z) dx' \right]$$

(0.21)

$$= F_x(0, y, z) + \int_0^x (-\nabla \cdot \mathbf{F} + \partial_{x'} F_{x'}) dx'$$

(0.22)

$$= F_x(0, y, z) + F_x(x, y, z) - F_x(0, y, z)$$

(0.23)

$$= F_x(x, y, z)$$

(0.24)

$$(\nabla \times \mathbf{A})_y = -\partial_x \left[ \int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx' \right] + \partial_z(0)$$

(0.25)

$$= F_y(x, y, z)$$

(0.26)

$$(\nabla \times \mathbf{A})_z = \partial_x \int_0^x F_z(x', y, z) dx' - \partial_y(0)$$

(0.27)

$$= F_z(x, y, z)$$

For the divergence

(0.28)

$$\nabla \cdot \mathbf{A} = \partial_x(0) + \partial_y \int_0^x F_z(x', y, z) dx' + \partial_z \left[ \int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx' \right]$$

This isn't zero in general.

For a specific example, consider

(0.29)

$$\mathbf{F} = y\hat{\mathbf{x}} + z\hat{\mathbf{y}} + x\hat{\mathbf{z}}$$

Then

$$(0.30) \quad A_x = 0$$

$$(0.31) \quad A_y = \int_0^x F_z(x', y, z) dx'$$

$$(0.32) \quad = \int_0^x x' dx'$$

$$(0.33) \quad = \frac{x^2}{2}$$

$$(0.34) \quad A_z = \int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx'$$

$$(0.35) \quad = \int_0^y y' dy' - \int_0^x z dx'$$

$$(0.36) \quad = \frac{y^2}{2} - xz$$

By direct calculation

$$(0.37) \quad (\nabla \times \mathbf{A})_x = \partial_y A_z - \partial_z A_y$$

$$(0.38) \quad = y = F_x$$

$$(0.39) \quad (\nabla \times \mathbf{A})_y = \partial_z A_x - \partial_x A_z$$

$$(0.40) \quad = z = F_y$$

$$(0.41) \quad (\nabla \times \mathbf{A})_z = \partial_x A_y - \partial_y A_x$$

$$(0.42) \quad = x = F_z$$