

DIVERGENCELESS VECTOR FIELD AS A CURL

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.30.

This is more mathematics than physics, but it relates to the magnetic vector potential so here we go. The vector potential \mathbf{A} is defined so that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\nabla \cdot \mathbf{B} = 0$. We'd like to prove that, in general, any divergenceless vector field \mathbf{F} can be written as the curl of another vector field \mathbf{A} . The curl $\mathbf{F} = \nabla \times \mathbf{A}$ has three components:

$$\partial_y A_z - \partial_z A_y = F_x \quad (1)$$

$$\partial_x A_z - \partial_z A_x = -F_y \quad (2)$$

$$\partial_x A_y - \partial_y A_x = F_z \quad (3)$$

Thus in principle, we have 3 coupled PDEs to solve, but we can get a solution by starting with the assumption that $A_x = 0$ (this isn't the only solution, of course, but it does give a solution). Then

$$\partial_x A_z = -F_y \quad (4)$$

$$A_z = - \int F_y(x', y, z) dx' + G(y, z) \quad (5)$$

$$\partial_x A_y = F_z \quad (6)$$

$$A_y = \int F_z(x', y, z) dx' + H(y, z) \quad (7)$$

where G and H are functions of integration; since the integrals are with respect to x , the 'constants' of integration can be functions of y and z .

We can now plug these into the first component of the curl above:

$$\begin{aligned} \partial_y A_z - \partial_z A_y &= - \int \partial_z F_y(x', y, z) dx' + \partial_y G(y, z) - \int \partial_y F_z(x', y, z) dx' - \partial_z H(y, z) \end{aligned} \quad (8)$$

$$= \int (-\nabla \cdot \mathbf{F} + \partial_{x'} F_{x'}) dx' + \partial_y G(y, z) - \partial_z H(y, z) \quad (9)$$

If we now want the result at a particular point (x, y, z) , we can introduce limits on the integral, and also use the requirement that $\nabla \cdot \mathbf{F} = 0$:

$$F_x(x, y, z) = \int_0^x \partial_{x'} F_{x'} dx' + \partial_y G(y, z) - \partial_z H(y, z) \quad (10)$$

$$= F_x(x, y, z) - F_x(0, y, z) + \partial_y G(y, z) - \partial_z H(y, z) \quad (11)$$

$$0 = -F_x(0, y, z) + \partial_y G(y, z) - \partial_z H(y, z) \quad (12)$$

At this point we can proceed in various ways, since the functions G and H have between them only this one condition. One option is to choose

$$G(y, z) = \int_0^y F_x(0, y', z) dy' \quad (13)$$

$$H(y, z) = 0 \quad (14)$$

With this option, we get

$$A_z = \int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx' \quad (15)$$

$$A_y = \int_0^x F_z(x', y, z) dx' \quad (16)$$

$$A_x = 0 \quad (17)$$

We could equally as well have chosen

$$G(y, z) = 0 \quad (18)$$

$$H(y, z) = - \int_0^z F_x(0, y, z') dz' \quad (19)$$

or some combination of the two.

Using the first option, we can check the curl.

$$(\nabla \times \mathbf{A})_x = \partial_y \left[\int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx' \right] - \partial_z \left[\int_0^x F_z(x', y, z) dx' \right] \quad (20)$$

$$= F_x(0, y, z) + \int_0^x (-\nabla \cdot \mathbf{F} + \partial_{x'} F_{x'}) dx' \quad (21)$$

$$= F_x(0, y, z) + F_x(x, y, z) - F_x(0, y, z) \quad (22)$$

$$= F_x(x, y, z) \quad (23)$$

$$(\nabla \times \mathbf{A})_y = -\partial_x \left[\int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx' \right] + \partial_z(0) \quad (24)$$

$$= F_y(x, y, z) \quad (25)$$

$$(\nabla \times \mathbf{A})_z = \partial_x \int_0^x F_z(x', y, z) dx' - \partial_y(0) \quad (26)$$

$$= F_z(x, y, z) \quad (27)$$

For the divergence

$$\nabla \cdot \mathbf{A} = \partial_x(0) + \partial_y \int_0^x F_z(x', y, z) dx' + \partial_z \left[\int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx' \right] \quad (28)$$

This isn't zero in general.

For a specific example, consider

$$\mathbf{F} = y\hat{\mathbf{x}} + z\hat{\mathbf{y}} + x\hat{\mathbf{z}} \quad (29)$$

Then

$$A_x = 0 \quad (30)$$

$$A_y = \int_0^x F_z(x', y, z) dx' \quad (31)$$

$$= \int_0^x x' dx' \quad (32)$$

$$= \frac{x^2}{2} \quad (33)$$

$$A_z = \int_0^y F_x(0, y', z) dy' - \int_0^x F_y(x', y, z) dx' \quad (34)$$

$$= \int_0^y y' dy' - \int_0^x z dx' \quad (35)$$

$$= \frac{y^2}{2} - xz \quad (36)$$

By direct calculation

$$(\nabla \times \mathbf{A})_x = \partial_y A_z - \partial_z A_y \quad (37)$$

$$= y = F_x \quad (38)$$

$$(\nabla \times \mathbf{A})_y = \partial_z A_x - \partial_x A_z \quad (39)$$

$$= z = F_y \quad (40)$$

$$(\nabla \times \mathbf{A})_z = \partial_x A_y - \partial_y A_x \quad (41)$$

$$= x = F_z \quad (42)$$