

MAGNETIC DIPOLE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.33.

In much the same way as the electric potential, we can write the magnetic potential as a multipole expansion. Because the magnetic potential is a vector, the expansion is a series of vector integrals rather than volume integrals.

The most general form of the vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (1)$$

For a steady line current I , this becomes a line integral around the current loop:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{l}' \quad (2)$$

We can expand the integrand in terms of Legendre polynomials in the same way as for the electric potential:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta'}} \quad (3)$$

$$= \frac{1}{r} \frac{1}{\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r} \cos \theta'}} \quad (4)$$

$$= \frac{1}{r} \sum_{n=0}^{\infty} P_n(\cos \theta') \left(\frac{r'}{r}\right)^n \quad (5)$$

The potential becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint P_n(\cos \theta') r'^n d\mathbf{l}' \quad (6)$$

The first three terms in this sum are called the monopole, dipole and quadrupole terms. The monopole term is

$$\mathbf{A}_0 = \frac{\mu_0 I}{4\pi r} \oint d\mathbf{l}' = 0 \quad (7)$$

since integration of the vector line element around any closed loop is zero. For the dipole:

$$\mathbf{A}_1 = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos \theta' d\mathbf{l}' \quad (8)$$

Since θ' is the angle between \mathbf{r}' and \mathbf{r} , we can write this as

$$\mathbf{A}_1 = \frac{\mu_0 I}{4\pi r^2} \oint \mathbf{r}' \cdot \hat{\mathbf{r}} d\mathbf{l}' \quad (9)$$

Since $\hat{\mathbf{r}}$ is a constant as far as the integral is concerned, we can write the integral in terms of the vector area \mathbf{a} enclosed by the loop.

$$\mathbf{A}_1 = \frac{\mu_0 I}{4\pi r^2} \mathbf{a} \times \hat{\mathbf{r}} \quad (10)$$

The quantity

$$\mathbf{m} \equiv I \mathbf{a} \quad (11)$$

is defined as the *magnetic dipole moment* of the current loop, so the dipole potential is

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi r^2} \mathbf{m} \times \hat{\mathbf{r}} \quad (12)$$

The magnetic field due to a dipole is

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1 \quad (13)$$

$$= \frac{\mu_0}{4\pi} \nabla \times \left(\mathbf{m} \times \frac{\hat{\mathbf{r}}}{r^2} \right) \quad (14)$$

We can use a vector identity to expand the curl, and use the fact that \mathbf{m} is a constant for a given current loop, so its derivatives are all zero. We get

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} \left[\mathbf{m} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) - (\mathbf{m} \cdot \nabla) \frac{\hat{\mathbf{r}}}{r^2} \right] \quad (15)$$

We can now use

$$\hat{\mathbf{r}} = \frac{1}{r} (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \quad (16)$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad (17)$$

Therefore

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = \partial_x \frac{x}{r^3} + \partial_y \frac{y}{r^3} + \partial_z \frac{z}{r^3} \quad (18)$$

$$= \left(-\frac{3}{2} \frac{2x^2}{r^5} + \frac{1}{r^3} \right) + \left(-\frac{3}{2} \frac{2y^2}{r^5} + \frac{1}{r^3} \right) + \left(-\frac{3}{2} \frac{2z^2}{r^5} + \frac{1}{r^3} \right) \quad (19)$$

$$= 0 \quad (20)$$

$$(\mathbf{m} \cdot \nabla) \frac{\hat{\mathbf{r}}}{r^2} = m_x \left[\left(-\frac{3}{2} \frac{2x^2}{r^5} + \frac{1}{r^3} \right) \hat{\mathbf{x}} - \frac{3}{2} \frac{2xy}{r^5} \hat{\mathbf{y}} - \frac{3}{2} \frac{2xz}{r^5} \hat{\mathbf{z}} \right] + m_y [\dots] + m_z [\dots] \quad (21)$$

$$= -3 \frac{xm_x}{r^5} \mathbf{r} + \frac{m_x}{r^3} \hat{\mathbf{x}} - 3 \frac{ym_y}{r^5} \mathbf{r} + \frac{m_y}{r^3} \hat{\mathbf{y}} - 3 \frac{zm_z}{r^5} \mathbf{r} + \frac{m_z}{r^3} \hat{\mathbf{z}} \quad (22)$$

$$= \frac{1}{r^3} [-3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} + \mathbf{m}] \quad (23)$$

Putting this all together, we get

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}] \quad (24)$$

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