MAGNETIC DIPOLE MOMENT OF A CURRENT LOOP

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.37.

Earlier we worked out the magnetic field at the centre of a square loop (side length w) of current which lies in the xy plane with its centre at the origin. We can generalize this a bit by working out the field at any point on the z axis. Start with the Biot-Savart law for linear currents:

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0 I}{4\pi}\int\frac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1}$$

Now consider the edge that extends from $y=-w/2$ to $y=+w/2$ at $x=+w/2$. Along this edge we have

$$\begin{aligned}
\mathbf{r}-\mathbf{r}' &= -x'\hat{\mathbf{x}}-y'\hat{\mathbf{y}}+z\hat{\mathbf{z}} &(2)\\
&= -\frac{w}{2}\hat{\mathbf{x}}-y'\hat{\mathbf{y}}+z\hat{\mathbf{z}} &(3)\\
d\mathbf{l}' &= dy'\hat{\mathbf{y}} &(4)\\
d\mathbf{l}'\times\left(\mathbf{r}-\mathbf{r}'\right) &= \left(z\hat{\mathbf{x}}+\frac{w}{2}\hat{\mathbf{z}}\right)dy' &(5)
\end{aligned}$$

The integral is then

$$\begin{aligned}
\mathbf{B}_1 &= \frac{\mu_0 I}{4\pi}\left(z\hat{\mathbf{x}}+\frac{w}{2}\hat{\mathbf{z}}\right)\int_{-w/2}^{w/2}\frac{dy'}{\left(\frac{w^2}{4}+y'^2+z^2\right)^{3/2}} &(6)\\
&= \frac{\mu_0 I}{4\pi}\left(z\hat{\mathbf{x}}+\frac{w}{2}\hat{\mathbf{z}}\right)\frac{8w}{\sqrt{2w^2+4z^2}\,(w^2+4z^2)} &(7)
\end{aligned}$$

By symmetry, the opposite edge at $x=-w/2$ will contribute the same z component and the opposite x component. Similarly, the two edges at $y=\pm w/2$ will contribute two more z components with the y components cancelling. Thus the total field is

$$\mathbf{B} = 4\frac{\mu_0 I}{4\pi}\frac{w}{2}\frac{8w}{\sqrt{2w^2+4z^2}\left(w^2+4z^2\right)}\hat{\mathbf{z}} \tag{8}$$

$$= \frac{4w^2\mu_0 I}{\pi\sqrt{2w^2+4z^2}\left(w^2+4z^2\right)}\hat{\mathbf{z}} \tag{9}$$

As a check, when $z=0$ this reduces to

$$\mathbf{B}(0) = \frac{2\sqrt{2}\mu_0 I}{\pi w}\hat{\mathbf{z}} \tag{10}$$

This matches the earlier post (where $w=2R$).

The dipole moment of the square is $\mathbf{m} = I\mathbf{a} = w^2\hat{\mathbf{z}}$, so the dipole component of the field is

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi z^3}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{m}\right] \tag{11}$$

$$= \frac{\mu_0 I}{4\pi z^3}\left(3w^2-w^2\right)\hat{\mathbf{z}} \tag{12}$$

$$= \frac{\mu_0 w^2 I}{2\pi z^3}\hat{\mathbf{z}} \tag{13}$$

If we take $z \gg w$ in the exact formula, it reduces to

$$\mathbf{B} \rightarrow \frac{4w^2\mu_0 I}{\pi\sqrt{4z^2}\left(4z^2\right)}\hat{\mathbf{z}} \tag{14}$$

$$= \frac{\mu_0 w^2 I}{2\pi z^3}\hat{\mathbf{z}} \tag{15}$$

Thus the dipole term is a good approximation for large distances.