

MAGNETIC DIPOLE MOMENT OF A CURRENT LOOP

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.37.

Earlier we worked out the magnetic field at the centre of a square loop (side length w) of current which lies in the xy plane with its centre at the origin. We can generalize this a bit by working out the field at any point on the z axis. Start with the Biot-Savart law for linear currents:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \quad (1)$$

Now consider the edge that extends from $y = -w/2$ to $y = +w/2$ at $x = +w/2$. Along this edge we have

$$\mathbf{r} - \mathbf{r}' = -x'\hat{\mathbf{x}} - y'\hat{\mathbf{y}} + z\hat{\mathbf{z}} \quad (2)$$

$$= -\frac{w}{2}\hat{\mathbf{x}} - y'\hat{\mathbf{y}} + z\hat{\mathbf{z}} \quad (3)$$

$$d\mathbf{l}' = dy'\hat{\mathbf{y}} \quad (4)$$

$$d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}') = \left(z\hat{\mathbf{x}} + \frac{w}{2}\hat{\mathbf{z}}\right) dy' \quad (5)$$

The integral is then

$$\mathbf{B}_1 = \frac{\mu_0 I}{4\pi} \left(z\hat{\mathbf{x}} + \frac{w}{2}\hat{\mathbf{z}}\right) \int_{-w/2}^{w/2} \frac{dy'}{\left(\frac{w^2}{4} + y'^2 + z^2\right)^{3/2}} \quad (6)$$

$$= \frac{\mu_0 I}{4\pi} \left(z\hat{\mathbf{x}} + \frac{w}{2}\hat{\mathbf{z}}\right) \frac{8w}{\sqrt{2w^2 + 4z^2}(w^2 + 4z^2)} \quad (7)$$

By symmetry, the opposite edge at $x = -w/2$ will contribute the same z component and the opposite x component. Similarly, the two edges at $y = \pm w/2$ will contribute two more z components with the y components cancelling. Thus the total field is

$$\mathbf{B} = 4 \frac{\mu_0 I w}{4\pi} \frac{8w}{2 \sqrt{2w^2 + 4z^2} (w^2 + 4z^2)} \hat{\mathbf{z}} \quad (8)$$

$$= \frac{4w^2 \mu_0 I}{\pi \sqrt{2w^2 + 4z^2} (w^2 + 4z^2)} \hat{\mathbf{z}} \quad (9)$$

As a check, when $z = 0$ this reduces to

$$\mathbf{B}(0) = \frac{2\sqrt{2}\mu_0 I}{\pi w} \hat{\mathbf{z}} \quad (10)$$

This matches the earlier post (where $w = 2R$).

The dipole moment of the square is $\mathbf{m} = I\mathbf{a} = w^2 \hat{\mathbf{z}}$, so the dipole component of the field is

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi z^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}] \quad (11)$$

$$= \frac{\mu_0 I}{4\pi z^3} (3w^2 - w^2) \hat{\mathbf{z}} \quad (12)$$

$$= \frac{\mu_0 w^2 I}{2\pi z^3} \hat{\mathbf{z}} \quad (13)$$

If we take $z \gg w$ in the exact formula, it reduces to

$$\mathbf{B} \rightarrow \frac{4w^2 \mu_0 I}{\pi \sqrt{4z^2} (4z^2)} \hat{\mathbf{z}} \quad (14)$$

$$= \frac{\mu_0 w^2 I}{2\pi z^3} \hat{\mathbf{z}} \quad (15)$$

Thus the dipole term is a good approximation for large distances.