

MAGNETIC MONOPOLE: FORCE ON A CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.43.

We'll have a look at a bit of fantasy physics, in the sense that it deals with something (magnetic monopoles) that are believed not to exist. Suppose they did exist, and that a 'magnetic charge' q_m produced a magnetic field according to

$$(0.1) \quad \mathbf{B} = \frac{\mu_0 q_m}{4\pi r^2} \hat{\mathbf{r}}$$

where \mathbf{r} is the vector position from the magnetic monopole. Then a particle of mass m with an ordinary charge q moving with velocity \mathbf{v} would experience a force:

$$(0.2) \quad \mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

$$(0.3) \quad \frac{d\mathbf{v}}{dt} = \frac{\mu_0 q q_m}{4\pi m r^2} \mathbf{v} \times \hat{\mathbf{r}}$$

Thus the acceleration is perpendicular to \mathbf{v} which means that the magnitude $|\mathbf{v}|$ is a constant. The argument is the same as that used for a centripetal force:

$$(0.4) \quad \frac{dv^2}{dt} = \frac{d(\mathbf{v} \cdot \mathbf{v})}{dt}$$

$$(0.5) \quad = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}$$

$$(0.6) \quad = 0$$

Thus v^2 and hence v is a constant.

We now introduce a rather cryptic quantity

$$(0.7) \quad \mathbf{Q} \equiv m\mathbf{r} \times \mathbf{v} - \frac{\mu_0 q q_m}{4\pi} \hat{\mathbf{r}}$$

This too is a constant, as we can see by taking its time derivative:

$$(0.8) \quad \frac{d\mathbf{Q}}{dt} = m\mathbf{v} \times \mathbf{v} + m\mathbf{r} \times \frac{d\mathbf{v}}{dt} - \frac{\mu_0 qq_m}{4\pi} \frac{d\hat{\mathbf{r}}}{dt}$$

The first term is zero, and we can substitute from above for $\frac{d\mathbf{v}}{dt}$ in the second. The third term is a bit special though. The derivative of $\hat{\mathbf{r}}$ is *not* zero, since the direction of the radius vector changes as the particle moves. We can work it out as follows:

$$(0.9) \quad \frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right)$$

$$(0.10) \quad = \frac{\mathbf{v}}{r} - \frac{1}{r^2} \frac{dr}{dt} \mathbf{r}$$

$$(0.11) \quad = \frac{\mathbf{v}}{r} - \frac{1}{r} \frac{dr}{dt} \hat{\mathbf{r}}$$

The trick here is to notice that $\frac{dr}{dt}$ is *not* in general equal to v . This is because $\frac{dr}{dt}$ measures the rate of change of distance from the origin, so if the particle moves, for example, in a circular orbit, $\frac{dr}{dt} = 0$ while $v \neq 0$. So what is it? It measures the rate of change of radial position, in other words, the component of velocity parallel to \mathbf{r} , which is given by $\frac{dr}{dt} = \mathbf{v} \cdot \hat{\mathbf{r}}$. Plugging all this in, we get

$$(0.12) \quad \frac{d\mathbf{Q}}{dt} = \frac{\mu_0 qq_m}{4\pi} \left[\frac{1}{r^2} \mathbf{r} \times (\mathbf{v} \times \hat{\mathbf{r}}) - \frac{1}{r} (\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}) \right]$$

$$(0.13) \quad = \frac{\mu_0 qq_m}{4\pi r} [\hat{\mathbf{r}} \times (\mathbf{v} \times \hat{\mathbf{r}}) - (\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}})]$$

$$(0.14) \quad = \frac{\mu_0 qq_m}{4\pi r} [\mathbf{v}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{v} \cdot \hat{\mathbf{r}}) - (\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}})]$$

$$(0.15) \quad = 0$$

where we used the vector identity for a triple product in line 3 to expand the first term.

Since \mathbf{Q} is constant, we can introduce spherical coordinates with the z axis along the \mathbf{Q} direction. By taking dot products of \mathbf{Q} with each of the unit vectors, we can derive some useful properties.

$$(0.16) \quad \mathbf{Q} \cdot \hat{\phi} = m\hat{\phi} \cdot (\mathbf{r} \times \mathbf{v})$$

$$(0.17) \quad = mr\mathbf{v} \cdot (\hat{\phi} \times \hat{\mathbf{r}})$$

$$(0.18) \quad = mr\mathbf{v} \cdot \hat{\theta}$$

where we've used another vector identity in line 2.

However, since \mathbf{Q} is along the vertical axis and $\hat{\phi}$ is always horizontal, $\mathbf{Q} \cdot \hat{\phi} = 0$. Therefore $\mathbf{v} \cdot \hat{\theta} = 0$ as well. To see what this means, we can write out \mathbf{v} as the derivative of the linear path of the particle. In spherical coordinates, a line element is

$$(0.19) \quad d\mathbf{l} = dr\hat{\mathbf{r}} + r d\theta\hat{\boldsymbol{\theta}} + r \sin\theta d\phi\hat{\boldsymbol{\phi}}$$

The velocity is

$$(0.20) \quad \mathbf{v} = \frac{d\mathbf{l}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

Therefore

$$(0.21) \quad \mathbf{v} \cdot \hat{\boldsymbol{\theta}} = r\frac{d\theta}{dt} = 0$$

which means that θ is a constant.

Next, consider $\mathbf{Q} \cdot \hat{\mathbf{r}} = |\mathbf{Q}| \cos\theta$ (since \mathbf{Q} is along the z axis). From the definition of \mathbf{Q} we get

$$(0.22) \quad |\mathbf{Q}| \cos\theta = -\frac{\mu_0 qq_m}{4\pi}$$

$$(0.23) \quad |\mathbf{Q}| = -\frac{\mu_0 qq_m}{4\pi \cos\theta}$$

Notice that this formula requires the sign of $\cos\theta$ to be opposite to that of qq_m , since $|\mathbf{Q}| > 0$. This means that \mathbf{Q} may point in the $\pm z$ direction so that this formula works out with the right sign. We don't actually use the direction of \mathbf{Q} (that is, whether it points up or down; we do, of course, use the fact that it is parallel to the z axis) in any of the calculations, so this isn't particularly important, but it's worth noting.

Finally, we consider

$$(0.24) \quad \mathbf{Q} \cdot \hat{\boldsymbol{\theta}} = mr\hat{\boldsymbol{\theta}} \cdot (\mathbf{r} \times \mathbf{v})$$

$$(0.25) \quad = mr\mathbf{v} \cdot (\hat{\boldsymbol{\theta}} \times \hat{\mathbf{r}})$$

$$(0.26) \quad = -mr\mathbf{v} \cdot \hat{\boldsymbol{\phi}}$$

Using 0.20 we get

$$(0.27) \quad -mr\mathbf{v} \cdot \hat{\phi} = -mr^2 \sin \theta \frac{d\phi}{dt}$$

Since the angle between the z axis and $\hat{\theta}$ is $\theta + \frac{\pi}{2}$, we have

$$(0.28) \quad \mathbf{Q} \cdot \hat{\theta} = -|\mathbf{Q}| \sin \theta$$

$$(0.29) \quad = \frac{\mu_0 q q_m \sin \theta}{4\pi \cos \theta}$$

Since $\sin \theta \geq 0$ for $0 \leq \theta \leq \pi$, the same sign relation between $\cos \theta$ and $q q_m$ holds here as well.

Therefore we have

$$(0.30) \quad -mr^2 \sin \theta \frac{d\phi}{dt} = \frac{\mu_0 q q_m \sin \theta}{4\pi \cos \theta}$$

$$(0.31) \quad \frac{d\phi}{dt} = -\frac{\mu_0 q q_m}{4\pi m r^2 \cos \theta}$$

$$(0.32) \quad = \frac{k}{r^2}$$

$$(0.33) \quad k \equiv -\frac{\mu_0 q q_m}{4\pi m \cos \theta}$$

We can eliminate the time by calculating (since $\frac{d\theta}{dt} = 0$)

$$(0.34) \quad v^2 = r^2 \sin^2 \theta \dot{\phi}^2 + \dot{r}^2$$

$$(0.35) \quad = r^2 \sin^2 \theta \frac{k^2}{r^4} + \left(\frac{dr}{d\phi} \dot{\phi} \right)^2$$

$$(0.36) \quad = r^2 \sin^2 \theta \frac{k^2}{r^4} + \left(\frac{dr}{d\phi} \right)^2 \frac{k^2}{r^4}$$

$$(0.37) \quad \frac{dr}{d\phi} = \sqrt{\frac{v^2 r^4}{k^2} - r^2 \sin^2 \theta}$$

Since v and θ are constants, this is a differential equation in r and ϕ only. We can integrate it to get

$$(0.38) \quad -\frac{1}{\sin \theta} \arctan \left(\frac{k \sin \theta}{\sqrt{v^2 r^2 - k^2 \sin^2 \theta}} \right) = \phi + \phi_0$$

$$(0.39) \quad r = \pm \frac{k \sin \theta}{v \sin ([\phi + \phi_0] \sin \theta)}$$