

MAGNETIC MONOPOLE: FORCE ON A CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.43.

We'll have a look at a bit of fantasy physics, in the sense that it deals with something (magnetic monopoles) that are believed not to exist. Suppose they did exist, and that a 'magnetic charge' q_m produced a magnetic field according to

$$\mathbf{B} = \frac{\mu_0 q_m}{4\pi r^2} \hat{\mathbf{r}} \quad (1)$$

where \mathbf{r} is the vector position from the magnetic monopole. Then a particle of mass m with an ordinary charge q moving with velocity \mathbf{v} would experience a force:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \quad (2)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\mu_0 q q_m}{4\pi m r^2} \mathbf{v} \times \hat{\mathbf{r}} \quad (3)$$

Thus the acceleration is perpendicular to \mathbf{v} which means that the magnitude $|\mathbf{v}|$ is a constant. The argument is the same as that used for a centripetal force:

$$\frac{dv^2}{dt} = \frac{d(\mathbf{v} \cdot \mathbf{v})}{dt} \quad (4)$$

$$= 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \quad (5)$$

$$= 0 \quad (6)$$

Thus v^2 and hence v is a constant.

We now introduce a rather cryptic quantity

$$\mathbf{Q} \equiv m\mathbf{r} \times \mathbf{v} - \frac{\mu_0 q q_m}{4\pi} \hat{\mathbf{r}} \quad (7)$$

This too is a constant, as we can see by taking its time derivative:

$$\frac{d\mathbf{Q}}{dt} = m\mathbf{v} \times \mathbf{v} + m\mathbf{r} \times \frac{d\mathbf{v}}{dt} - \frac{\mu_0 q q_m}{4\pi} \frac{d\hat{\mathbf{r}}}{dt} \quad (8)$$

The first term is zero, and we can substitute from above for $\frac{d\mathbf{v}}{dt}$ in the second. The third term is a bit special though. The derivative of $\hat{\mathbf{r}}$ is *not* zero, since the direction of the radius vector changes as the particle moves. We can work it out as follows:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right) \quad (9)$$

$$= \frac{\mathbf{v}}{r} - \frac{1}{r^2} \frac{dr}{dt} \mathbf{r} \quad (10)$$

$$= \frac{\mathbf{v}}{r} - \frac{1}{r} \frac{dr}{dt} \hat{\mathbf{r}} \quad (11)$$

The trick here is to notice that $\frac{dr}{dt}$ is *not* in general equal to v . This is because $\frac{dr}{dt}$ measures the rate of change of distance from the origin, so if the particle moves, for example, in a circular orbit, $\frac{dr}{dt} = 0$ while $v \neq 0$. So what is it? It measures the rate of change of radial position, in other words, the component of velocity parallel to \mathbf{r} , which is given by $\frac{dr}{dt} = \mathbf{v} \cdot \hat{\mathbf{r}}$. Plugging all this in, we get

$$\frac{d\mathbf{Q}}{dt} = \frac{\mu_0 qq_m}{4\pi} \left[\frac{1}{r^2} \mathbf{r} \times (\mathbf{v} \times \hat{\mathbf{r}}) - \frac{1}{r} (\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}) \right] \quad (12)$$

$$= \frac{\mu_0 qq_m}{4\pi r} [\hat{\mathbf{r}} \times (\mathbf{v} \times \hat{\mathbf{r}}) - (\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}})] \quad (13)$$

$$= \frac{\mu_0 qq_m}{4\pi r} [\mathbf{v} (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}} (\mathbf{v} \cdot \hat{\mathbf{r}}) - (\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}})] \quad (14)$$

$$= 0 \quad (15)$$

where we used the vector identity for a triple product in line 3 to expand the first term.

Since \mathbf{Q} is constant, we can introduce spherical coordinates with the z axis along the \mathbf{Q} direction. By taking dot products of \mathbf{Q} with each of the unit vectors, we can derive some useful properties.

$$\mathbf{Q} \cdot \hat{\phi} = m\hat{\phi} \cdot (\mathbf{r} \times \mathbf{v}) \quad (16)$$

$$= mr\mathbf{v} \cdot (\hat{\phi} \times \hat{\mathbf{r}}) \quad (17)$$

$$= mr\mathbf{v} \cdot \hat{\theta} \quad (18)$$

where we've used another vector identity in line 2.

However, since \mathbf{Q} is along the vertical axis and $\hat{\phi}$ is always horizontal, $\mathbf{Q} \cdot \hat{\phi} = 0$. Therefore $\mathbf{v} \cdot \hat{\theta} = 0$ as well. To see what this means, we can

write out \mathbf{v} as the derivative of the linear path of the particle. In spherical coordinates, a line element is

$$d\mathbf{l} = dr\hat{\mathbf{r}} + r d\theta\hat{\boldsymbol{\theta}} + r \sin\theta d\phi\hat{\boldsymbol{\phi}} \quad (19)$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{l}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r \sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}} \quad (20)$$

Therefore

$$\mathbf{v} \cdot \hat{\boldsymbol{\theta}} = r\frac{d\theta}{dt} = 0 \quad (21)$$

which means that θ is a constant.

Next, consider $\mathbf{Q} \cdot \hat{\mathbf{r}} = |\mathbf{Q}| \cos\theta$ (since \mathbf{Q} is along the z axis). From the definition of \mathbf{Q} we get

$$|\mathbf{Q}| \cos\theta = -\frac{\mu_0 q q_m}{4\pi} \quad (22)$$

$$|\mathbf{Q}| = -\frac{\mu_0 q q_m}{4\pi \cos\theta} \quad (23)$$

Notice that this formula requires the sign of $\cos\theta$ to be opposite to that of $q q_m$, since $|\mathbf{Q}| > 0$. This means that \mathbf{Q} may point in the $\pm z$ direction so that this formula works out with the right sign. We don't actually use the direction of \mathbf{Q} (that is, whether it points up or down; we do, of course, use the fact that it is parallel to the z axis) in any of the calculations, so this isn't particularly important, but it's worth noting.

Finally, we consider

$$\mathbf{Q} \cdot \hat{\boldsymbol{\theta}} = mr\hat{\boldsymbol{\theta}} \cdot (\mathbf{r} \times \mathbf{v}) \quad (24)$$

$$= mr\mathbf{v} \cdot (\hat{\boldsymbol{\theta}} \times \hat{\mathbf{r}}) \quad (25)$$

$$= -mr\mathbf{v} \cdot \hat{\boldsymbol{\phi}} \quad (26)$$

Using 20 we get

$$-mr\mathbf{v} \cdot \hat{\boldsymbol{\phi}} = -mr^2 \sin\theta \frac{d\phi}{dt} \quad (27)$$

Since the angle between the z axis and $\hat{\boldsymbol{\theta}}$ is $\theta + \frac{\pi}{2}$, we have

$$\mathbf{Q} \cdot \hat{\boldsymbol{\theta}} = -|\mathbf{Q}| \sin\theta \quad (28)$$

$$= \frac{\mu_0 q q_m \sin\theta}{4\pi \cos\theta} \quad (29)$$

Since $\sin \theta \geq 0$ for $0 \leq \theta \leq \pi$, the same sign relation between $\cos \theta$ and qq_m holds here as well.

Therefore we have

$$-mr^2 \sin \theta \frac{d\phi}{dt} = \frac{\mu_0 qq_m \sin \theta}{4\pi \cos \theta} \quad (30)$$

$$\frac{d\phi}{dt} = -\frac{\mu_0 qq_m}{4\pi mr^2 \cos \theta} \quad (31)$$

$$= \frac{k}{r^2} \quad (32)$$

$$k \equiv -\frac{\mu_0 qq_m}{4\pi m \cos \theta} \quad (33)$$

We can eliminate the time by calculating (since $\frac{d\theta}{dt} = 0$)

$$v^2 = r^2 \sin^2 \theta \dot{\phi}^2 + \dot{r}^2 \quad (34)$$

$$= r^2 \sin^2 \theta \frac{k^2}{r^4} + \left(\frac{dr}{d\phi} \dot{\phi} \right)^2 \quad (35)$$

$$= r^2 \sin^2 \theta \frac{k^2}{r^4} + \left(\frac{dr}{d\phi} \right)^2 \frac{k^2}{r^4} \quad (36)$$

$$\frac{dr}{d\phi} = \sqrt{\frac{v^2 r^4}{k^2} - r^2 \sin^2 \theta} \quad (37)$$

Since v and θ are constants, this is a differential equation in r and ϕ only. We can integrate it to get

$$-\frac{1}{\sin \theta} \arctan \left(\frac{k \sin \theta}{\sqrt{v^2 r^2 - k^2 \sin^2 \theta}} \right) = \phi + \phi_0 \quad (38)$$

$$r = \pm \frac{k \sin \theta}{v \sin([\phi + \phi_0] \sin \theta)} \quad (39)$$