MAGNETIC MONOPOLE: FORCE ON A CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.43.

We'll have a look at a bit of fantasy physics, in the sense that it deals with something (magnetic monopoles) that are believed not to exist. Suppose they did exist, and that a 'magnetic charge' q_m produced a magnetic field according to

$$\mathbf{B}=\frac{\mu_0 q_m}{4\pi r^2}\hat{\mathbf{r}} \tag{1}$$

where $\mathbf{r}$ is the vector position from the magnetic monopole. Then a particle of mass m with an ordinary charge q moving with velocity $\mathbf{v}$ would experience a force:

$$\begin{aligned}
\mathbf{F} &= q\mathbf{v}\times\mathbf{B} && (2)\\
\frac{d\mathbf{v}}{dt} &= \frac{\mu_0 q q_m}{4\pi m r^2}\mathbf{v}\times\hat{\mathbf{r}} && (3)
\end{aligned}$$

Thus the acceleration is perpendicular to $\mathbf{v}$ which means that the magnitude $|\mathbf{v}|$ is a constant. The argument is the same as that used for a centripetal force:

$$\begin{aligned}
\frac{dv^2}{dt} &= \frac{d(\mathbf{v}\cdot\mathbf{v})}{dt} && (4)\\
&= 2\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} && (5)\\
&= 0 && (6)
\end{aligned}$$

Thus v^2 and hence v is a constant.

We now introduce a rather cryptic quantity

$$\mathbf{Q}\equiv m\mathbf{r}\times\mathbf{v}-\frac{\mu_0 q q_m}{4\pi}\hat{\mathbf{r}} \tag{7}$$

This too is a constant, as we can see by taking its time derivative:

$$\frac{d\mathbf{Q}}{dt}=m\mathbf{v}\times\mathbf{v}+m\mathbf{r}\times\frac{d\mathbf{v}}{dt}-\frac{\mu_0 q q_m}{4\pi}\frac{d\hat{\mathbf{r}}}{dt} \tag{8}$$

The first term is zero, and we can substitute from above for $\frac{d\mathbf{v}}{dt}$ in the second. The third term is a bit special though. The derivative of $\hat{\mathbf{r}}$ is *not* zero, since the direction of the radius vector changes as the particle moves. We can work it out as follows:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \tag{9}$$

$$= \frac{\mathbf{v}}{r} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r} \tag{10}$$

$$= \frac{\mathbf{v}}{r} - \frac{1}{r}\frac{dr}{dt}\hat{\mathbf{r}} \tag{11}$$

The trick here is to notice that $\frac{dr}{dt}$ is *not* in general equal to v. This is because $\frac{dr}{dt}$ measures the rate of change of distance from the origin, so if the particle moves, for example, in a circular orbit, $\frac{dr}{dt} = 0$ while $v \neq 0$. So what is it? It measures the rate of change of radial position, in other words, the component of velocity parallel to $\mathbf{r}$, which is given by $\frac{dr}{dt} = \mathbf{v}\cdot\hat{\mathbf{r}}$. Plugging all this in, we get

$$\frac{d\mathbf{Q}}{dt} = \frac{\mu_0 q q_m}{4\pi}\left[\frac{1}{r^2}\mathbf{r}\times(\mathbf{v}\times\hat{\mathbf{r}}) - \frac{1}{r}(\mathbf{v}-(\mathbf{v}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right] \tag{12}$$

$$= \frac{\mu_0 q q_m}{4\pi r}\left[\hat{\mathbf{r}}\times(\mathbf{v}\times\hat{\mathbf{r}}) - (\mathbf{v}-(\mathbf{v}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right] \tag{13}$$

$$= \frac{\mu_0 q q_m}{4\pi r}\left[\mathbf{v}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{v}\cdot\hat{\mathbf{r}}) - (\mathbf{v}-(\mathbf{v}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})\right] \tag{14}$$

$$= 0 \tag{15}$$

where we used the vector identity for a triple product in line 3 to expand the first term.

Since $\mathbf{Q}$ is constant, we can introduce spherical coordinates with the z axis along the $\mathbf{Q}$ direction. By taking dot products of $\mathbf{Q}$ with each of the unit vectors, we can derive some useful properties.

$$\mathbf{Q}\cdot\hat{\phi} = m\hat{\phi}\cdot(\mathbf{r}\times\mathbf{v}) \tag{16}$$

$$= mr\mathbf{v}\cdot\left(\hat{\phi}\times\hat{\mathbf{r}}\right) \tag{17}$$

$$= mr\mathbf{v}\cdot\hat{\theta} \tag{18}$$

where we've used another vector identity in line 2.

However, since $\mathbf{Q}$ is along the vertical axis and $\hat{\phi}$ is always horizontal, $\mathbf{Q}\cdot\hat{\phi}=0$. Therefore $\mathbf{v}\cdot\hat{\theta}=0$ as well. To see what this means, we can write out $\mathbf{v}$ as the derivative of the linear path of the particle. In spherical coordinates, a line element is

$$d\mathbf{l}=dr\hat{\mathbf{r}}+rd\theta\hat{\theta}+r\sin\theta d\phi\hat{\phi} \tag{19}$$

The velocity is

$$\mathbf{v}=\frac{d\mathbf{l}}{dt}=\frac{dr}{dt}\hat{\mathbf{r}}+r\frac{d\theta}{dt}\hat{\theta}+r\sin\theta\frac{d\phi}{dt}\hat{\phi} \tag{20}$$

Therefore

$$\mathbf{v}\cdot\hat{\theta}=r\frac{d\theta}{dt}=0 \tag{21}$$

which means that θ is a constant.

Next, consider $\mathbf{Q}\cdot\hat{\mathbf{r}}=|\mathbf{Q}|\cos\theta$ (since $\mathbf{Q}$ is along the z axis). From the definition of $\mathbf{Q}$ we get

$$|\mathbf{Q}|\cos\theta = -\frac{\mu_0 q q_m}{4\pi} \tag{22}$$

$$|\mathbf{Q}| = -\frac{\mu_0 q q_m}{4\pi\cos\theta} \tag{23}$$

Notice that this formula requires the sign of $\cos\theta$ to be opposite to that of qq_m, since $|\mathbf{Q}|>0$. This means that $\mathbf{Q}$ may point in the $\pm z$ direction so that this formula works out with the right sign. We don't actually use the direction of $\mathbf{Q}$ (that is, whether it points up or down; we do, of course, use the fact that it is parallel to the z axis) in any of the calculations, so this isn't particularly important, but it's worth noting.

Finally, we consider

$$\mathbf{Q}\cdot\hat{\theta} = mr\hat{\theta}\cdot(\mathbf{r}\times\mathbf{v}) \tag{24}$$

$$= mr\mathbf{v}\cdot\left(\hat{\theta}\times\hat{\mathbf{r}}\right) \tag{25}$$

$$= -mr\mathbf{v}\cdot\hat{\phi} \tag{26}$$

Using 20 we get

$$-mr\mathbf{v}\cdot\hat{\phi}=-mr^2\sin\theta\frac{d\phi}{dt} \tag{27}$$

Since the angle between the z axis and $\hat{\theta}$ is $\theta+\frac{\pi}{2}$, we have

$$\mathbf{Q}\cdot\hat{\theta} = -|\mathbf{Q}|\sin\theta \tag{28}$$

$$= \frac{\mu_0 q q_m \sin\theta}{4\pi\cos\theta} \tag{29}$$

Since $\sin\theta\geq 0$ for $0\leq\theta\leq\pi$, the same sign relation between $\cos\theta$ and qq_m holds here as well.

Therefore we have

$$-mr^2\sin\theta\frac{d\phi}{dt} = \frac{\mu_0 q q_m \sin\theta}{4\pi\cos\theta} \tag{30}$$

$$\frac{d\phi}{dt} = -\frac{\mu_0 q q_m}{4\pi m r^2\cos\theta} \tag{31}$$

$$= \frac{k}{r^2} \tag{32}$$

$$k \equiv -\frac{\mu_0 q q_m}{4\pi m\cos\theta} \tag{33}$$

We can eliminate the time by calculating (since $\frac{d\theta}{dt}=0$)

$$v^2 = r^2\sin^2\theta\dot{\phi}^2+\dot{r}^2 \tag{34}$$

$$= r^2\sin^2\theta\frac{k^2}{r^4}+\left(\frac{dr}{d\phi}\dot{\phi}\right)^2 \tag{35}$$

$$= r^2\sin^2\theta\frac{k^2}{r^4}+\left(\frac{dr}{d\phi}\right)^2\frac{k^2}{r^4} \tag{36}$$

$$\frac{dr}{d\phi} = \sqrt{\frac{v^2r^4}{k^2}-r^2\sin^2\theta} \tag{37}$$

Since v and θ are constants, this is a differential equation in r and ϕ only. We can integrate it to get

$$-\frac{1}{\sin\theta}\arctan\left(\frac{k\sin\theta}{\sqrt{v^2r^2-k^2\sin^2\theta}}\right)=\phi+\phi_0 \tag{38}$$

$$r=\pm\frac{k\sin\theta}{v\sin\left([\phi+\phi_0]\sin\theta\right)} \tag{39}$$