SOLENOID FIELD FROM BIOT-SAVART LAW

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.44.

We've used Ampère's law to find the field inside and outside a solenoid. The fundamental formula for finding the magnetic field due to a current is the Biot-Savart law, so it should be possible to work out the solenoid field from that as well. Since we treat the current in a solenoid as cylindrical surface current, the form of the Biot-Savart law to use is

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{K}\times\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3}da' \tag{1}$$

where $\mathbf{K}$ is the surface current density. For a solenoid with n turns per unit length carrying a current I, $K=nI$.

Although the natural coordinates to use are cylindrical, I find it easier to set the problem up in rectangular coordinates and then convert to cylindrical later on. To define the problem, put the axis of the solenoid on the z axis and place the observation point $\mathbf{r}$ on the x axis. The source point $\mathbf{r}'$ lies on the solenoid. If the radius of the solenoid is R, then

$$\mathbf{r}' = x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}} \tag{2}$$

$$x^2+y^2 = R^2 \tag{3}$$

$$\mathbf{r} = r\hat{\mathbf{x}} \tag{4}$$

$$\mathbf{r}-\mathbf{r}' = (r-x)\hat{\mathbf{x}}-y\hat{\mathbf{y}}-z\hat{\mathbf{z}} \tag{5}$$

Since $\mathbf{K}$ points around the circumference of the solenoid,

$$\mathbf{K}=nI\hat{\phi} \tag{6}$$

$$=nI\left(-\sin\phi\hat{\mathbf{x}}+\cos\phi\hat{\mathbf{y}}\right) \tag{7}$$

$$\mathbf{K}\times\left(\mathbf{r}-\mathbf{r}'\right)=nI\left[-z\cos\phi\hat{\mathbf{x}}-z\sin\phi\hat{\mathbf{y}}+\left(y\sin\phi-(r-x)\cos\phi\right)\hat{\mathbf{z}}\right] \tag{8}$$

From the symmetry of the setup, if we replace z by $-z$ in the source point $\mathbf{r}'$, $\mathbf{K}$ remains unchanged, but the x and y components of $\mathbf{K}\times\left(\mathbf{r}-\mathbf{r}'\right)$ change sign. Thus these components cancel out and the net field must lie in the z direction, so we can restrict our attention to that from now on.

We can now make the conversion to cylindrical coordinates using

$$\begin{aligned} y &= R\sin\phi && (9) \\ x &= R\cos\phi && (10) \\ da' &= Rd\phi dz && (11) \end{aligned}$$

We get

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0 nI}{4\pi}\hat{\mathbf{z}}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{R\left(\sin^2\phi+\cos^2\phi\right)-r\cos\phi}{\left[(r-x)^2+y^2+z^2\right]^{3/2}}Rdzd\phi && (12) \\ &= \frac{\mu_0 nI}{4\pi}\hat{\mathbf{z}}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{R^2-rR\cos\phi}{\left[R^2+r^2-2rR\cos\phi+z^2\right]^{3/2}}dzd\phi && (13) \end{aligned}$$

At this point it's important to do the integrals in the right order. Attempting to do the ϕ integral first leads to a mess containing elliptic functions. If we do the z integral first, we get

$$\mathbf{B} = \frac{2\mu_0 nI}{4\pi}\hat{\mathbf{z}}\int_0^{2\pi}\frac{R^2-rR\cos\phi}{R^2+r^2-2rR\cos\phi}d\phi \tag{14}$$

Using software, this integral comes out to

$$\mathbf{B} = \begin{cases} \mu_0 nI\hat{\mathbf{z}} & r<R \\ 0 & r>R \end{cases} \tag{15}$$

This reproduces, after a lot of effort, the result obtained from Ampère's law.