

## SOLENOID FIELD FROM BIOT-SAVART LAW

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.44.

We've used Ampère's law to find the field inside and outside a solenoid. The fundamental formula for finding the magnetic field due to a current is the Biot-Savart law, so it should be possible to work out the solenoid field from that as well. Since we treat the current in a solenoid as cylindrical surface current, the form of the Biot-Savart law to use is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} da' \quad (1)$$

where  $\mathbf{K}$  is the surface current density. For a solenoid with  $n$  turns per unit length carrying a current  $I$ ,  $K = nI$ .

Although the natural coordinates to use are cylindrical, I find it easier to set the problem up in rectangular coordinates and then convert to cylindrical later on. To define the problem, put the axis of the solenoid on the  $z$  axis and place the observation point  $\mathbf{r}$  on the  $x$  axis. The source point  $\mathbf{r}'$  lies on the solenoid. If the radius of the solenoid is  $R$ , then

$$\mathbf{r}' = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \quad (2)$$

$$x^2 + y^2 = R^2 \quad (3)$$

$$\mathbf{r} = r\hat{\mathbf{x}} \quad (4)$$

$$\mathbf{r} - \mathbf{r}' = (r - x)\hat{\mathbf{x}} - y\hat{\mathbf{y}} - z\hat{\mathbf{z}} \quad (5)$$

Since  $\mathbf{K}$  points around the circumference of the solenoid,

$$\mathbf{K} = nI\hat{\phi} \quad (6)$$

$$= nI(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}) \quad (7)$$

$$\mathbf{K} \times (\mathbf{r} - \mathbf{r}') = nI[-z\cos\phi\hat{\mathbf{x}} - z\sin\phi\hat{\mathbf{y}} + (y\sin\phi - (r-x)\cos\phi)\hat{\mathbf{z}}] \quad (8)$$

From the symmetry of the setup, if we replace  $z$  by  $-z$  in the source point  $\mathbf{r}'$ ,  $\mathbf{K}$  remains unchanged, but the  $x$  and  $y$  components of  $\mathbf{K} \times (\mathbf{r} - \mathbf{r}')$  change sign. Thus these components cancel out and the net field must lie in the  $z$  direction, so we can restrict our attention to that from now on.

We can now make the conversion to cylindrical coordinates using

$$y = R \sin \phi \quad (9)$$

$$x = R \cos \phi \quad (10)$$

$$da' = R d\phi dz \quad (11)$$

We get

$$\mathbf{B} = \frac{\mu_0 n I}{4\pi} \hat{\mathbf{z}} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{R (\sin^2 \phi + \cos^2 \phi) - r \cos \phi}{[(r-x)^2 + y^2 + z^2]^{3/2}} R dz d\phi \quad (12)$$

$$= \frac{\mu_0 n I}{4\pi} \hat{\mathbf{z}} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{R^2 - r R \cos \phi}{[R^2 + r^2 - 2r R \cos \phi + z^2]^{3/2}} dz d\phi \quad (13)$$

At this point it's important to do the integrals in the right order. Attempting to do the  $\phi$  integral first leads to a mess containing elliptic functions. If we do the  $z$  integral first, we get

$$\mathbf{B} = \frac{2\mu_0 n I}{4\pi} \hat{\mathbf{z}} \int_0^{2\pi} \frac{R^2 - r R \cos \phi}{R^2 + r^2 - 2r R \cos \phi} d\phi \quad (14)$$

Using software, this integral comes out to

$$\mathbf{B} = \begin{cases} \mu_0 n I \hat{\mathbf{z}} & r < R \\ 0 & r > R \end{cases} \quad (15)$$

This reproduces, after a lot of effort, the result obtained from Ampère's law.