

SOLENOID FIELD FROM BIOT-SAVART LAW

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.44.

We've used Ampère's law to find the field inside and outside a solenoid. The fundamental formula for finding the magnetic field due to a current is the Biot-Savart law, so it should be possible to work out the solenoid field from that as well. Since we treat the current in a solenoid as cylindrical surface current, the form of the Biot-Savart law to use is

$$(1) \quad \mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} da'$$

where \mathbf{K} is the surface current density. For a solenoid with n turns per unit length carrying a current I , $K = nI$.

Although the natural coordinates to use are cylindrical, I find it easier to set the problem up in rectangular coordinates and then convert to cylindrical later on. To define the problem, put the axis of the solenoid on the z axis and place the observation point \mathbf{r} on the x axis. The source point \mathbf{r}' lies on the solenoid. If the radius of the solenoid is R , then

$$(2) \quad \mathbf{r}' = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$(3) \quad x^2 + y^2 = R^2$$

$$(4) \quad \mathbf{r} = r\hat{\mathbf{x}}$$

$$(5) \quad \mathbf{r} - \mathbf{r}' = (r - x)\hat{\mathbf{x}} - y\hat{\mathbf{y}} - z\hat{\mathbf{z}}$$

Since \mathbf{K} points around the circumference of the solenoid,

$$(6) \quad \mathbf{K} = nI\hat{\phi}$$

$$(7) \quad = nI(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})$$

$$(8) \quad \mathbf{K} \times (\mathbf{r} - \mathbf{r}') = nI[-z\cos\phi\hat{\mathbf{x}} - z\sin\phi\hat{\mathbf{y}} + (y\sin\phi - (r-x)\cos\phi)\hat{\mathbf{z}}]$$

From the symmetry of the setup, if we replace z by $-z$ in the source point \mathbf{r}' , \mathbf{K} remains unchanged, but the x and y components of $\mathbf{K} \times (\mathbf{r} - \mathbf{r}')$ change sign. Thus these components cancel out and the net field must lie in the z direction, so we can restrict our attention to that from now on.

We can now make the conversion to cylindrical coordinates using

$$(9) \quad y = R \sin \phi$$

$$(10) \quad x = R \cos \phi$$

$$(11) \quad da' = R d\phi dz$$

We get

$$(12) \quad \mathbf{B} = \frac{\mu_0 n I}{4\pi} \hat{\mathbf{z}} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{R(\sin^2 \phi + \cos^2 \phi) - r \cos \phi}{[(r-x)^2 + y^2 + z^2]^{3/2}} R dz d\phi$$

$$(13) \quad = \frac{\mu_0 n I}{4\pi} \hat{\mathbf{z}} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{R^2 - rR \cos \phi}{[R^2 + r^2 - 2rR \cos \phi + z^2]^{3/2}} dz d\phi$$

At this point it's important to do the integrals in the right order. Attempting to do the ϕ integral first leads to a mess containing elliptic functions. If we do the z integral first, we get

$$(14) \quad \mathbf{B} = \frac{2\mu_0 n I}{4\pi} \hat{\mathbf{z}} \int_0^{2\pi} \frac{R^2 - rR \cos \phi}{R^2 + r^2 - 2rR \cos \phi} d\phi$$

Using software, this integral comes out to

$$(15) \quad \mathbf{B} = \begin{cases} \mu_0 n I \hat{\mathbf{z}} & r < R \\ 0 & r > R \end{cases}$$

This reproduces, after a lot of effort, the result obtained from Ampère's law.