MAGNETIC FIELD OF A SEMI-CIRCULAR CURRENT LOOP

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.45.

Another example of finding the magnetic field due to a current using the Biot-Savart law.

The system this time consists of a semi-circular wire of radius R in the xy plane, extending from $\theta=-\pi$ to $\theta=0$ and carrying current I in a counter-clockwise direction. The problem is to find the magnetic field on a point on the upper semi-circle of radius R extending from $\theta=0$ to $\theta=\pi$. (There is no wire or current on this upper semi-circle; we're just interested in finding the field there.) Let the angle of the observation point be $\theta=\alpha$.

The Biot-Savart law for a line current is

$$\mathbf{B}=\frac{\mu_0 I}{4\pi}\int\frac{d\mathbf{l}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1}$$

Since $d\mathbf{l}'$ lies on the semi-circular wire

$$d\mathbf{l}'=Rd\theta\left(-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}\right) \tag{2}$$

For a given observation point $\mathbf{r}$ and source point $\mathbf{r}'$

$$
\begin{aligned}
\mathbf{r} &= x\hat{\mathbf{x}}+y\hat{\mathbf{y}} && (3)\\
&= R\cos\alpha\hat{\mathbf{x}}+R\sin\alpha\hat{\mathbf{y}} && (4)\\
\mathbf{r}' &= x'\hat{\mathbf{x}}+y'\hat{\mathbf{y}} && (5)\\
&= R\cos\theta\hat{\mathbf{x}}+R\sin\theta\hat{\mathbf{y}} && (6)\\
\mathbf{r}-\mathbf{r}' &= \left(x-x'\right)\hat{\mathbf{x}}+\left(y-y'\right)\hat{\mathbf{y}} && (7)\\
&= R\left(\cos\alpha-\cos\theta\right)\hat{\mathbf{x}}+R\left(\sin\alpha-\sin\theta\right)\hat{\mathbf{y}} && (8)\\
\left|\mathbf{r}-\mathbf{r}'\right| &= R\sqrt{\left(\cos\alpha-\cos\theta\right)^2+\left(\sin\alpha-\sin\theta\right)^2} && (9)\\
&= R\sqrt{2\left(1-\sin\theta\sin\alpha-\cos\theta\cos\alpha\right)} && (10)\\
&= \sqrt{2}R\sqrt{1-\cos\left(\alpha-\theta\right)} && (11)\\
d\mathbf{l}'\times\left(\mathbf{r}-\mathbf{r}'\right) &= -R^2d\theta\left[\sin\theta\left(\sin\alpha-\sin\theta\right)+\cos\theta\left(\cos\alpha-\cos\theta\right)\right]\hat{\mathbf{z}} && (12)\\
&= R^2d\theta\left[1-\sin\theta\sin\alpha-\cos\theta\cos\alpha\right]\hat{\mathbf{z}} && (13)\\
&= R^2d\theta\left[1-\cos\left(\alpha-\theta\right)\right]\hat{\mathbf{z}} && (14)
\end{aligned}
$$

We can now write out the integral:

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0 IR^2}{4\pi 2^{3/2}R^3}\hat{\mathbf{z}}\int\frac{\left[1-\cos\left(\alpha-\theta\right)\right]d\theta}{\left(1-\cos\left(\alpha-\theta\right)\right)^{3/2}} && (15)\\
&= \frac{\mu_0 I}{2^{3/2}4\pi R}\hat{\mathbf{z}}\int\frac{d\theta}{\sqrt{1-\cos\left(\alpha-\theta\right)}} && (16)
\end{aligned}
$$

Using software, this integral comes out to

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{2^{3/2}4\pi R}\hat{\mathbf{z}}\sqrt{2}\frac{\left(1-\cos^2\left[\frac{1}{2}\left(\alpha-\theta\right)\right]\right)\tanh^{-1}\left(\cos\left[\frac{1}{2}\left(\alpha-\theta\right)\right]\right)}{\sqrt{1-\cos^2\left[\frac{1}{2}\left(\alpha-\theta\right)\right]}\sin\left[\frac{1}{2}\left(\alpha-\theta\right)\right]} && (17)\\
&\frac{\mu_0 I}{2^{3/2}4\pi R}\hat{\mathbf{z}}\sqrt{2}\tanh^{-1}\left(\cos\left[\frac{1}{2}\left(\alpha-\theta\right)\right]\right) && (18)\\
&= \frac{\mu_0 I}{16\pi R}\hat{\mathbf{z}}\ln\left[\frac{1+\cos\left[\frac{1}{2}\left(\alpha-\theta\right)\right]}{1-\cos\left[\frac{1}{2}\left(\alpha-\theta\right)\right]}\right] && (19)
\end{aligned}
$$

where we've used the identity $\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.

Up to now, I've purposely left off the limits of integration. If we use the limits as stated in the question above, we get

$$\mathbf{B} = \frac{\mu_0 I}{16\pi R}\hat{\mathbf{z}}\ln\left[\frac{1+\cos\left[\frac{1}{2}(\alpha-\theta)\right]}{1-\cos\left[\frac{1}{2}(\alpha-\theta)\right]}\right]\Bigg|_{-\pi}^{0} \tag{20}$$

$$= \frac{\mu_0 I}{16\pi R}\hat{\mathbf{z}}\ln\left[\frac{1+\cos\left(\frac{\alpha}{2}\right)}{1-\cos\left(\frac{\alpha}{2}\right)}\frac{1-\cos\left[\frac{1}{2}(\alpha+\pi)\right]}{1+\cos\left[\frac{1}{2}(\alpha+\pi)\right]}\right] \tag{21}$$

$$= \frac{\mu_0 I}{16\pi R}\hat{\mathbf{z}}\ln\left[\frac{\tan^2\left(\frac{1}{4}(\alpha+\pi)\right)}{\tan^2\left(\frac{\alpha}{4}\right)}\right] \tag{22}$$

$$= \frac{\mu_0 I}{8\pi R}\hat{\mathbf{z}}\ln\left[\left|\frac{\tan\left(\frac{1}{4}(\alpha+\pi)\right)}{\tan\left(\frac{\alpha}{4}\right)}\right|\right] \tag{23}$$

where we've used the trig identities $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$.

This matches the answer given in the question in Griffiths (where his θ is my α). However, if we choose the limits to be $\theta = \pi$ to $\theta = 2\pi$, we get

$$\mathbf{B} = \frac{\mu_0 I}{16\pi R}\hat{\mathbf{z}}\ln\left[\frac{1-\cos\left(\frac{\alpha}{2}\right)}{1+\cos\left(\frac{\alpha}{2}\right)}\frac{1-\cos\left[\frac{1}{2}(\alpha-\pi)\right]}{1+\cos\left[\frac{1}{2}(\alpha-\pi)\right]}\right] \tag{24}$$

$$= \frac{\mu_0 I}{8\pi R}\hat{\mathbf{z}}\ln\left[\left|\tan\left(\frac{\alpha}{4}\right)\tan\left(\frac{1}{4}(\alpha-\pi)\right)\right|\right] \tag{25}$$

Since $\left|\tan\left(x-\frac{\pi}{2}\right)\right| = \left|\frac{1}{\tan x}\right|$ this comes out to

$$\mathbf{B} = -\frac{\mu_0 I}{8\pi R}\hat{\mathbf{z}}\ln\left[\left|\frac{\tan\left(\frac{1}{4}(\alpha+\pi)\right)}{\tan\left(\frac{\alpha}{4}\right)}\right|\right] \tag{26}$$

That is, by merely changing the limits of integration by adding 2π, we've reversed the direction of **B**. This clearly doesn't make sense, since the integrand $\frac{1}{\sqrt{1-\cos(\alpha-\theta)}}$ is identical under this change, so it seems that we need to take the absolute value of the answer. We can see how this errant sign crept in by looking back at the original integral solution above. There, we cancelled off the cos and sin terms that multiplied the arctanh, but the square root term could be positive or negative (something which the software didn't tell us), so presumably we need to select the right sign in order for the direction of the field to come out right. The correct direction can be determined from the right hand rule and for the direction of current assumed, **B** must point in the $+z$ direction.