

## MAGNETIC FIELDS OF SPINNING DISK AND SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.47.

Another application of the magnetic field due to a circular current loop of radius  $R$  and current  $I$  a distance  $z$  above the centre of the loop (as given by Griffiths example 5.6):

$$B = \frac{\mu_0 I R^2}{2} \frac{1}{[R^2 + z^2]^{3/2}} \quad (1)$$

We start with a spinning disk with surface charge density  $\sigma$ . We can treat this as a collection of concentric current loops, with the current at radius  $r$  given by

$$dI(r) = \sigma r \omega dr \quad (2)$$

where  $\omega$  is the angular velocity. The field of the spinning disk is then

$$B(z) = \frac{\mu_0 \sigma \omega}{2} \int_0^R \frac{r^3}{(r^2 + z^2)^{3/2}} dr \quad (3)$$

$$= \frac{\mu_0 \sigma \omega}{2} \left[ \frac{2z^2 + R^2}{\sqrt{R^2 + z^2}} - 2z \right] \quad (4)$$

Now we can work out the field due to a spinning solid sphere of charge density  $\rho$  as a collection of spinning disks. We fix our observation point  $z$ , and place the origin at the centre of the sphere. A given disk within the sphere has a radius given by  $R \sin \theta$  for a particular value of  $\theta$ . The distance of this disk from the observation point is  $z - R \cos \theta$  and the charge density in the disk is  $\rho R d\theta$ . It might seem that the way to proceed is to replace  $\sigma \rightarrow \rho R d\theta$ ,  $R \rightarrow R \sin \theta$  and  $z \rightarrow z - R \cos \theta$  in the formula for the spinning disk and then integrate over  $\theta$  from 0 to  $\pi$ . That is, we try:

$$B(z) = \frac{\mu_0 \rho R \omega}{2} \int_0^\pi \left[ \frac{2(z - R \cos \theta)^2 + (R \sin \theta)^2}{\sqrt{(R \sin \theta)^2 + (z - R \cos \theta)^2}} - 2(z - R \cos \theta) \right] d\theta \quad (5)$$

Even after multiplying out the various squares and trying to simplify this expression, plugging it into software gives a complex sum of elliptic functions. It turns out that a better approach is to use an integral over the  $z$  coordinate of the disks. Let  $\zeta$  be the  $z$  coordinate of a given disk. Then the distance from the disk to the observation point is  $z - \zeta$ , the radius of the disk is  $\sqrt{R^2 - \zeta^2}$  and the charge density is  $\rho d\zeta$ . Now the appropriate substitutions into the formula for the disk's magnetic field are  $z \rightarrow z - \zeta$ ,  $R \rightarrow \sqrt{R^2 - \zeta^2}$  and  $\sigma \rightarrow \rho d\zeta$ , and the integral is over  $\zeta$  between  $-R$  and  $R$ :

$$B(z) = \frac{\mu_0 \rho \omega}{2} \int_{-R}^R \left[ \frac{2(z - \zeta)^2 + R^2 - \zeta^2}{\sqrt{R^2 - \zeta^2 + (z - \zeta)^2}} - 2(z - \zeta) \right] d\zeta \quad (6)$$

$$= \frac{\mu_0 \rho \omega}{2} \int_{-R}^R \left[ \frac{2z^2 + R^2 - 4z\zeta + \zeta^2}{\sqrt{R^2 + z^2 - 2z\zeta}} - 2(z - \zeta) \right] d\zeta \quad (7)$$

$$= \frac{2}{15} \frac{\mu_0 \rho \omega R^5}{z^3} \quad (8)$$

Software was used for the integral, but the first term in the integrand can be split into 3 (one for each term in the numerator) and either integrated directly or by parts, so it's not too hard even by hand. Note that we've assumed that  $z > R$ . If we take  $0 \leq z \leq R$  then we get contributions to the field from two different regions:  $0 \leq \zeta \leq z$  and  $z \leq \zeta \leq R$ . The field  $B_1$  from the inner region can be obtained from 7 by setting  $R = z$  and integrating from  $-z$  to  $z$

$$B_1 = \frac{\mu_0 \rho \omega}{2} \int_{-z}^z \left[ \frac{3z^2 - 4z\zeta + \zeta^2}{\sqrt{2z^2 - 2z\zeta}} - 2(z - \zeta) \right] d\zeta \quad (9)$$

$$= \frac{2}{15} \mu_0 \rho \omega z^2 \quad (10)$$

Each spherical shell of radius  $\zeta$  in the outer region contributes a uniform field of magnitude (as worked out in Griffiths's example 5.11):

$$B_{shell} = \frac{2}{3} \mu_0 \sigma \omega \zeta \quad (11)$$

ere  $\sigma$  is the surface charge density on the shell. A shell from our solid sphere of charge has a charge density of  $\sigma = \rho d\zeta$  so the field due to the outer region is

$$B_2 = \frac{2}{3}\mu_0\rho\omega \int_z^R \zeta d\zeta \quad (12)$$

$$= \frac{1}{3}\mu_0\rho\omega (R^2 - z^2) \quad (13)$$

The total field for  $z < R$  is then

$$B_{z < R} = B_1 + B_2 \quad (14)$$

$$= \frac{2}{15}\mu_0\rho\omega z^2 + \frac{1}{3}\mu_0\rho\omega (R^2 - z^2) \quad (15)$$

$$= \mu_0\rho\omega \left( \frac{R^2}{3} - \frac{z^2}{5} \right) \quad (16)$$

This agrees with our earlier calculation of the field inside a rotating sphere

$$\mathbf{B} = \frac{\mu_0\omega\rho}{3} \left[ \left( R_0^2 - \frac{3}{5}r^2 \right) \cos\theta \hat{\mathbf{r}} + \left( \frac{6}{5}r^2 - R_0^2 \right) \sin\theta \hat{\theta} \right] \quad (17)$$

To get the field along the  $z$  axis, we set  $\theta = 0$  and  $r = z$ , so we get the same formula.