MAGNETIC FIELDS OF SPINNING DISK AND SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.47.

Another application of the magnetic field due to a circular current loop of radius R and current I a distance z above the centre of the loop (as given by Griffiths example 5.6):

$$B=\frac{\mu_0 IR^2}{2}\frac{1}{\left[R^2+z^2\right]^{3/2}} \tag{1}$$

We start with a spinning disk with surface charge density σ. We can treat this as a collection of concentric current loops, with the current at radius r given by

$$dI(r)=\sigma r\omega dr \tag{2}$$

where ω is the angular velocity. The field of the spinning disk is then

$$B(z) = \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3}{\left(r^2+z^2\right)^{3/2}}dr \tag{3}$$

$$= \frac{\mu_0\sigma\omega}{2}\left[\frac{2z^2+R^2}{\sqrt{R^2+z^2}}-2z\right] \tag{4}$$

Now we can work out the field due to a spinning solid sphere of charge density ρ as a collection of spinning disks. We fix our observation point z, and place the origin at the centre of the sphere. A given disk within the sphere has a radius given by $R\sin\theta$ for a particular value of θ. The distance of this disk from the observation point is $z-R\cos\theta$ and the charge density in the disk is $\rho Rd\theta$. It might seem that the way to proceed is to replace $\sigma\to\rho Rd\theta$ $R\to R\sin\theta$ and $z\to z-R\cos\theta$ in the formula for the spinning disk and then integrate over θ from 0 to π. That is, we try:

(5)
$$B(z)=\frac{\mu_0\rho R\omega}{2}\int_0^{\pi}\left[\frac{2(z-R\cos\theta)^2+(R\sin\theta)^2}{\sqrt{(R\sin\theta)^2+(z-R\cos\theta)^2}}-2(z-R\cos\theta)\right]d\theta$$

Even after multiplying out the various squares and trying to simplify this expression, plugging it into software gives a complex sum of elliptic functions. It turns out that a better approach is to use an integral over the z coordinate of the disks. Let ζ be the z coordinate of a given disk. Then the distance from the disk to the observation point is $z-\zeta$, the radius of the disk is $\sqrt{R^2-\zeta^2}$ and the charge density is $\rho d\zeta$. Now the appropriate substitutions into the formula for the disk's magnetic field are $z\to z-\zeta$, $R\to\sqrt{R^2-\zeta^2}$ and $\sigma\to\rho d\zeta$, and the integral is over ζ between $-R$ and R:

$$B(z)=\frac{\mu_0\rho\omega}{2}\int_{-R}^{R}\left[\frac{2(z-\zeta)^2+R^2-\zeta^2}{\sqrt{R^2-\zeta^2+(z-\zeta)^2}}-2(z-\zeta)\right]d\zeta \tag{6}$$

$$=\frac{\mu_0\rho\omega}{2}\int_{-R}^{R}\left[\frac{2z^2+R^2-4z\zeta+\zeta^2}{\sqrt{R^2+z^2-2z\zeta}}-2(z-\zeta)\right]d\zeta \tag{7}$$

$$=\frac{2}{15}\frac{\mu_0\rho\omega R^5}{z^3} \tag{8}$$

Software was used for the integral, but the first term in the integrand can be split into 3 (one for each term in the numerator) and either integrated directly or by parts, so it's not too hard even by hand. Note that we've assumed that $z>R$. If we take $0\le z\le R$ then we get contributions to the field from two different regions: $0\le\zeta\le z$ and $z\le\zeta\le R$. The field B_1 from the inner region can be obtained from 7 by setting $R=z$ and integrating from $-z$ to z

$$B_1=\frac{\mu_0\rho\omega}{2}\int_{-z}^{z}\left[\frac{3z^2-4z\zeta+\zeta^2}{\sqrt{2z^2-2z\zeta}}-2(z-\zeta)\right]d\zeta \tag{9}$$

$$=\frac{2}{15}\mu_0\rho\omega z^2 \tag{10}$$

Each spherical shell of radius ζ in the outer region contributes a uniform field of magnitude (as worked out in Griffiths's example 5.11):

$$B_{shell} = \frac{2}{3}\mu_0 \sigma \omega \zeta \tag{11}$$

ere σ is the surface charge density on the shell. A shell from our solid sphere of charge has a charge density of $\sigma = \rho\, d\zeta$ so the field due to the outer region is

$$\begin{aligned} B_2 &= \frac{2}{3}\mu_0 \rho \omega \int_z^R \zeta\, d\zeta && (12) \\ &= \frac{1}{3}\mu_0 \rho \omega \left(R^2 - z^2\right) && (13) \end{aligned}$$

The total field for $z < R$ is then

$$\begin{aligned} B_{z<R} &= B_1 + B_2 && (14) \\ &= \frac{2}{15}\mu_0 \rho \omega z^2 + \frac{1}{3}\mu_0 \rho \omega \left(R^2 - z^2\right) && (15) \\ &= \mu_0 \rho \omega \left(\frac{R^2}{3} - \frac{z^2}{5}\right) && (16) \end{aligned}$$

This agrees with our earlier calculation of the field inside a rotating sphere

$$\mathbf{B} = \frac{\mu_0 \omega \rho}{3}\left[\left(R_0^2 - \frac{3}{5}r^2\right)\cos\theta\,\hat{\mathbf{r}} + \left(\frac{6}{5}r^2 - R_0^2\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right] \tag{17}$$

To get the field along the z axis, we set $\theta = 0$ and $r = z$, so we get the same formula.