

MAGNETIC FIELD OF CURRENT LOOP - OFF AXIS FIELD

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.48.

The magnetic field due to a circular current loop of radius R and current I a distance z above the centre of the loop is (as given by Griffiths example 5.6):

$$B = \frac{\mu_0 I R^2}{2} \frac{1}{[R^2 + z^2]^{3/2}} \quad (1)$$

The derivation of this was made easier by restricting our attention to points on the z axis. In principle, it's easy enough to find the field at any other point, but the integrals prove somewhat problematical. Suppose we take our observation point \mathbf{r} to be in the yz plane, so that

$$\mathbf{r} = y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \quad (2)$$

The loop lies in the xy plane, so

$$\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} \quad (3)$$

$$= R\cos\phi'\hat{\mathbf{x}} + R\sin\phi'\hat{\mathbf{y}} \quad (4)$$

$$\mathbf{r} - \mathbf{r}' = -R\cos\phi'\hat{\mathbf{x}} + (y - R\sin\phi')\hat{\mathbf{y}} + z\hat{\mathbf{z}} \quad (5)$$

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(R\cos\phi')^2 + (y - R\sin\phi')^2 + z^2} \quad (6)$$

$$= \sqrt{R^2 + r^2 - 2yR\sin\phi'} \quad (7)$$

The line element is

$$d\mathbf{l}' = [-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}] R d\phi' \quad (8)$$

$$d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}') = \left\{ z\cos\phi'\hat{\mathbf{x}} + z\sin\phi'\hat{\mathbf{y}} + \left[(R\sin\phi' - y)\sin\phi' + (R\cos\phi')^2 \right] \hat{\mathbf{z}} \right\} R d\phi' \quad (9)$$

$$= [z\cos\phi'\hat{\mathbf{x}} + z\sin\phi'\hat{\mathbf{y}} + (R - y\sin\phi')\hat{\mathbf{z}}] R d\phi' \quad (10)$$

The field is then, using the Biot-Savart formula:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\mathbf{Y}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \quad (11)$$

$$= \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{[z \cos \phi' \hat{\mathbf{x}} + z \sin \phi' \hat{\mathbf{y}} + (R - y \sin \phi') \hat{\mathbf{z}}] d\phi'}{(R^2 + r^2 - 2yR \sin \phi')^{3/2}} \quad (12)$$

The x component integrates easily to

$$B_x = \frac{\mu_0 I z}{4\pi y} \frac{1}{\sqrt{R^2 + r^2 - 2yR \sin \phi'}} \Big|_0^{2\pi} \quad (13)$$

$$= 0 \quad (14)$$

The fact that $B_x = 0$ isn't terribly surprising, since the symmetry of the problem makes that fairly obvious without doing any calculations.

The other two integrals are non-trivial, however. Plugging them into software we get a blizzard of elliptic functions with no obvious way of simplifying the result. We *can* check that these integrals reduce to the correct form when the observation point is on the z axis (that is, when $y = 0$). In that case we get

$$B_y = \frac{\mu_0 I R}{4\pi} \frac{z}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} \sin \phi' d\phi' = 0 \quad (15)$$

$$B_z = \frac{\mu_0 I}{4\pi} \frac{R^2}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi' = \frac{\mu_0 I R^2}{2} \frac{1}{[R^2 + z^2]^{3/2}} \quad (16)$$