

## MAGNETIC FIELD OF CURRENT LOOP - OFF AXIS FIELD

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.48.

The magnetic field due to a circular current loop of radius  $R$  and current  $I$  a distance  $z$  above the centre of the loop is (as given by Griffiths example 5.6):

$$(0.1) \quad B = \frac{\mu_0 I R^2}{2} \frac{1}{[R^2 + z^2]^{3/2}}$$

The derivation of this was made easier by restricting our attention to points on the  $z$  axis. In principle, it's easy enough to find the field at any other point, but the integrals prove somewhat problematical. Suppose we take our observation point  $\mathbf{r}$  to be in the  $yz$  plane, so that

$$(0.2) \quad \mathbf{r} = y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

The loop lies in the  $xy$  plane, so

$$(0.3) \quad \mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$$

$$(0.4) \quad = R \cos \phi' \hat{\mathbf{x}} + R \sin \phi' \hat{\mathbf{y}}$$

$$(0.5) \quad \mathbf{r} - \mathbf{r}' = -R \cos \phi' \hat{\mathbf{x}} + (y - R \sin \phi') \hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$(0.6) \quad |\mathbf{r} - \mathbf{r}'| = \sqrt{(R \cos \phi')^2 + (y - R \sin \phi')^2 + z^2}$$

$$(0.7) \quad = \sqrt{R^2 + r^2 - 2yR \sin \phi'}$$

The line element is

(0.8)

$$d\mathbf{l}' = [-\sin \phi' \hat{\mathbf{x}} + \cos \phi' \hat{\mathbf{y}}] R d\phi'$$

(0.9)

$$d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}') = \left\{ z \cos \phi' \hat{\mathbf{x}} + z \sin \phi' \hat{\mathbf{y}} + \left[ (R \sin \phi' - y) \sin \phi' + (R \cos \phi')^2 \right] \hat{\mathbf{z}} \right\} R d\phi'$$

(0.10)

$$= [z \cos \phi' \hat{\mathbf{x}} + z \sin \phi' \hat{\mathbf{y}} + (R - y \sin \phi') \hat{\mathbf{z}}] R d\phi'$$

The field is then, using the Biot-Savart formula:

$$(0.11) \quad \mathbf{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

$$(0.12) \quad = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{[z \cos \phi' \hat{\mathbf{x}} + z \sin \phi' \hat{\mathbf{y}} + (R - y \sin \phi') \hat{\mathbf{z}}] d\phi'}{(R^2 + r^2 - 2yR \sin \phi')^{3/2}}$$

The  $x$  component integrates easily to

$$(0.13) \quad B_x = \frac{\mu_0 I z}{4\pi y} \frac{1}{\sqrt{R^2 + r^2 - 2yR \sin \phi'}} \Big|_0^{2\pi}$$

$$(0.14) \quad = 0$$

The fact that  $B_x = 0$  isn't terribly surprising, since the symmetry of the problem makes that fairly obvious without doing any calculations.

The other two integrals are non-trivial, however. Plugging them into software we get a blizzard of elliptic functions with no obvious way of simplifying the result. We *can* check that these integrals reduce to the correct form when the observation point is on the  $z$  axis (that is, when  $y = 0$ ). In that case we get

$$(0.15) \quad B_y = \frac{\mu_0 I R}{4\pi} \frac{z}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} \sin \phi' d\phi' = 0$$

$$(0.16) \quad B_z = \frac{\mu_0 I}{4\pi} \frac{R^2}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi' = \frac{\mu_0 I R^2}{2} \frac{1}{[R^2 + z^2]^{3/2}}$$