MAGNETIC FIELD OF CURRENT LOOP - OFF AXIS FIELD

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.48.

The magnetic field due to a circular current loop of radius R and current I a distance z above the centre of the loop is (as given by Griffiths example 5.6):

$$B=\frac{\mu_0 IR^2}{2}\frac{1}{\left[R^2+z^2\right]^{3/2}} \tag{1}$$

The derivation of this was made easier by restricting our attention to points on the z axis. In principle, it's easy enough to find the field at any other point, but the integrals prove somewhat problematical. Suppose we take our observation point $\mathbf{r}$ to be in the yz plane, so that

$$\mathbf{r}=y\hat{\mathbf{y}}+z\hat{\mathbf{z}} \tag{2}$$

The loop lies in the xy plane, so

$$\begin{aligned}
\mathbf{r}' &= x'\hat{\mathbf{x}}+y'\hat{\mathbf{y}} && (3)\\
&= R\cos\phi'\hat{\mathbf{x}}+R\sin\phi'\hat{\mathbf{y}} && (4)\\
\mathbf{r}-\mathbf{r}' &= -R\cos\phi'\hat{\mathbf{x}}+\left(y-R\sin\phi'\right)\hat{\mathbf{y}}+z\hat{\mathbf{z}} && (5)\\
\left|\mathbf{r}-\mathbf{r}'\right| &= \sqrt{\left(R\cos\phi'\right)^2+\left(y-R\sin\phi'\right)^2+z^2} && (6)\\
&= \sqrt{R^2+r^2-2yR\sin\phi'} && (7)
\end{aligned}$$

The line element is

(8)
$$d\mathbf{l}' = \left[-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}}\right] R d\phi'$$
(9)
$$d\mathbf{l}' \times \left(\mathbf{r} - \mathbf{r}'\right) = \left\{z\cos\phi'\hat{\mathbf{x}} + z\sin\phi'\hat{\mathbf{y}} + \left[\left(R\sin\phi' - y\right)\sin\phi' + \left(R\cos\phi'\right)^2\right]\hat{\mathbf{z}}\right\} R d\phi'$$
(10)
$$= \left[z\cos\phi'\hat{\mathbf{x}} + z\sin\phi'\hat{\mathbf{y}} + \left(R - y\sin\phi'\right)\hat{\mathbf{z}}\right] R d\phi'$$

The field is then, using the Biot-Savart formula:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{11}$$

$$= \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{[z\cos\phi'\hat{\mathbf{x}} + z\sin\phi'\hat{\mathbf{y}} + (R - y\sin\phi')\hat{\mathbf{z}}]\,d\phi'}{(R^2 + r^2 - 2yR\sin\phi')^{3/2}} \tag{12}$$

The x component integrates easily to

$$B_x = \frac{\mu_0 I z}{4\pi y}\left.\frac{1}{\sqrt{R^2 + r^2 - 2yR\sin\phi'}}\right|_0^{2\pi} \tag{13}$$

$$= 0 \tag{14}$$

The fact that $B_x = 0$ isn't terribly surprising, since the symmetry of the problem makes that fairly obvious without doing any calculations.

The other two integrals are non-trivial, however. Plugging them into software we get a blizzard of elliptic functions with no obvious way of simplifying the result. We *can* check that these integrals reduce to the correct form when the observation point is on the z axis (that is, when $y = 0$). In that case we get

$$B_y = \frac{\mu_0 IR}{4\pi}\frac{z}{(R^2 + z^2)^{3/2}}\int_0^{2\pi}\sin\phi' d\phi' = 0 \tag{15}$$

$$B_z = \frac{\mu_0 I}{4\pi}\frac{R^2}{(R^2 + z^2)^{3/2}}\int_0^{2\pi} d\phi' = \frac{\mu_0 IR^2}{2}\frac{1}{[R^2 + z^2]^{3/2}} \tag{16}$$