FORCE BETWEEN CURRENT LOOPS: NEWTON'S THIRD LAW

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.49.

We can write the force on one current loop due to a second current loop in a symmetric form by combining the Biot-Savart law and Lorentz force law. The force on loop 1 due to loop 2 is

$$\mathbf{F}_1 = I_1 \oint d\mathbf{l}_1 \times \mathbf{B}_2 \tag{1}$$

The magnetic field produced by loop 2 at a point $\mathbf{r}_1$ on loop 1 is

$$\mathbf{B}_2 = \frac{\mu_0 I_2}{4\pi} \oint \frac{d\mathbf{l}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{2}$$

If we define $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$, we can substitute this field into the force integral to get

$$\begin{aligned}
\mathbf{F}_1 &= \frac{\mu_0 I_1 I_2}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \times [d\mathbf{l}_2 \times \mathbf{r}]}{r^3} && (3)\\
&= \frac{\mu_0 I_1 I_2}{4\pi} \oint \oint \frac{d\mathbf{l}_2 (d\mathbf{l}_1 \cdot \mathbf{r}) - \mathbf{r} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{r^3} && (4)\\
&= \frac{\mu_0 I_1 I_2}{4\pi} \oint \oint \frac{d\mathbf{l}_2 (d\mathbf{l}_1 \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{r^2} && (5)
\end{aligned}$$

where we've used a vector identity in line 2.

Consider now the term $d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. The $d\mathbf{l}_1$ is an increment along loop 1 while $\mathbf{r}_2$ is held constant. Thus $d\mathbf{l}_1$ is the change in $\mathbf{r}$ as we move a bit along loop 1. The dot product $d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$ is the component of this change that is parallel to $\mathbf{r}$; that is, it is the change in the magnitude (as opposed to the direction) of $\mathbf{r}$. We can therefore write

$$d\mathbf{l}_1 \cdot \hat{\mathbf{r}} = dr \tag{6}$$

The first term therefore becomes

$$\frac{\mu_0 I_1 I_2}{4\pi} \oint d\mathbf{l}_2 \oint \frac{dr}{r^2} \tag{7}$$

where we've taken $d\mathbf{l}_2$ outside the first integral since the point on loop 2 is held constant in the inner integral. The inner integral is zero, since we can split the closed loop into 2 parts, one from point **a** to point **b** and the other from **b** back to **a** by carrying along in the same direction around the loop. The first integral is

$$\int_{\mathbf{a}}^{\mathbf{b}} \frac{dr}{r^2} = \frac{1}{r(\mathbf{a})} - \frac{1}{r(\mathbf{b})} \tag{8}$$

That is, it depends only on the value of r at the endpoints and not on the path taken between them. The integral on the reverse path is therefore just the negative of the first integral, so the integral around the closed loop is zero. We therefore get for the force

$$\mathbf{F}_1 = -\frac{\mu_0 I_1 I_2}{4\pi} \oint \oint \frac{\hat{\mathbf{r}} (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{r^2} \tag{9}$$

If we interchange the loops to find the force $\mathbf{F}_2$ on loop 2 due to loop 1, everything is the same except that $\hat{\mathbf{r}}$ becomes $-\hat{\mathbf{r}}$, so $\mathbf{F}_2 = -\mathbf{F}_1$ which is an example of Newton's third law (equal and opposite forces).