MAGNETIC VECTOR POTENTIAL FROM MAGNETIC FIELD

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.51.

The magnetic vector potential can be used to find the field through the relation $\mathbf{B}=\nabla\times\mathbf{A}$, but there is no common formula for calculating the potential from the field. The potential is usually calculated from the current density via the formula

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{1}$$

One attempt at a formula for deriving $\mathbf{A}$ from $\mathbf{B}$ is

$$\mathbf{A}(\mathbf{r})=\int_{\mathcal{O}}^{\mathbf{r}}\mathbf{B}\times d\mathbf{l} \tag{2}$$

where the integral is a line integral over some path from the origin (which must be defined independently) to the observation point $\mathbf{r}$. For a constant field, this gives

$$\begin{aligned}\mathbf{A} &= \mathbf{B}\times\mathbf{r} &(3)\\ &= -\mathbf{r}\times\mathbf{B} &(4)\end{aligned}$$

This differs from the correct value by a factor of $\frac{1}{2}$, but there is a more serious problem in that the line integral isn't always independent of the path. To see this, consider the case of an infinite wire with a current I. The magnetic field circles the wire so in cylindrical coordinates

$$\mathbf{B}=\frac{\mu_0 I}{2\pi r}\hat{\phi} \tag{5}$$

If we integrate $\mathbf{B}$ around a closed square loop of side 1 that is perpendicular to the field, the contributions from the two edges perpendicular to the wire will cancel so we get

$$\oint\mathbf{B}\times d\mathbf{l}=\frac{\mu_0 I}{2\pi}\left(\frac{1}{a}-\frac{1}{a+1}\right) \tag{6}$$

where a is the distance of the near edge from the wire. This is clearly not zero, which it would have to be for the integral to be path independent.

Another attempt at a formula is

$$\mathbf{A}(\mathbf{r}) = -\mathbf{r}\times\int_0^1 \lambda\mathbf{B}(\lambda\mathbf{r})\,d\lambda \tag{7}$$

(This formula is stated in Griffiths's problem.)

For a constant field, this gives

$$\begin{aligned}\mathbf{A} &= -\mathbf{r}\times\mathbf{B}\int_0^1 \lambda d\lambda &(8)\\ &= -\frac{1}{2}\mathbf{r}\times\mathbf{B} &(9)\end{aligned}$$

which is correct.

In the case of the infinite wire, we have

$$\mathbf{B}(\lambda\mathbf{r}) = \frac{\mu_0 I}{2\pi(\lambda s)}\hat{\phi} \tag{10}$$

where s is the perpendicular distance from the wire to the point $\mathbf{r}$. Therefore

$$\begin{aligned}\int_0^1 \lambda\mathbf{B}(\lambda\mathbf{r})\,d\lambda &= \frac{\mu_0 I}{2\pi s}\hat{\phi}\int_0^1 d\lambda &(11)\\ &= \frac{\mu_0 I}{2\pi s}\hat{\phi} &(12)\end{aligned}$$

In cylindrical coordinates, the position vector $\mathbf{r} = s\hat{\mathbf{s}} + z\hat{\mathbf{z}}$ so

$$\begin{aligned}\mathbf{A} &= -\frac{\mu_0 I}{2\pi s}(s\hat{\mathbf{s}} + z\hat{\mathbf{z}})\times\hat{\phi} &(13)\\ &= \frac{\mu_0 I}{2\pi s}(z\hat{\mathbf{s}} - s\hat{\mathbf{z}}) &(14)\end{aligned}$$

In this case, $\nabla\cdot\mathbf{A} = 0$ and $\nabla\times\mathbf{A} = \mathbf{B}$ as can be checked by direct calculation, so this is a valid potential. However, in general, a potential calculated this way does not satisfy $\nabla\cdot\mathbf{A} = 0$. We have

$$\nabla \cdot \mathbf{A} = -\nabla \cdot \left[\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r})\, d\lambda \right] \tag{15}$$

$$= \mathbf{r} \cdot \int_0^1 \lambda \nabla \times \mathbf{B}(\lambda \mathbf{r})\, d\lambda - \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) \cdot \nabla \times \mathbf{r} d\lambda \tag{16}$$

$$= \mathbf{r} \cdot \int_0^1 \lambda^2 \mu_0 \mathbf{J}(\lambda \mathbf{r})\, d\lambda - 0 \tag{17}$$

where we've used Ampère's law and the chain rule in the first term, and the fact that $\nabla \times \mathbf{r} = 0$ in the second. Thus the only way this can be zero for all observation points $\mathbf{r}$ is for the integral to be zero, which effectively means that there can be no current. So this formula for the potential is not in general divergence-free.