

MAGNETIC VECTOR POTENTIAL FROM MAGNETIC FIELD

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.51.

The magnetic vector potential can be used to find the field through the relation $\mathbf{B} = \nabla \times \mathbf{A}$, but there is no common formula for calculating the potential from the field. The potential is usually calculated from the current density via the formula

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (1)$$

One attempt at a formula for deriving \mathbf{A} from \mathbf{B} is

$$\mathbf{A}(\mathbf{r}) = \int_O^{\mathbf{r}} \mathbf{B} \times d\mathbf{l} \quad (2)$$

where the integral is a line integral over some path from the origin (which must be defined independently) to the observation point \mathbf{r} . For a constant field, this gives

$$\mathbf{A} = \mathbf{B} \times \mathbf{r} \quad (3)$$

$$= -\mathbf{r} \times \mathbf{B} \quad (4)$$

This differs from the correct value by a factor of $\frac{1}{2}$, but there is a more serious problem in that the line integral isn't always independent of the path. To see this, consider the case of an infinite wire with a current I . The magnetic field circles the wire so in cylindrical coordinates

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi} \quad (5)$$

If we integrate \mathbf{B} around a closed square loop of side l that is perpendicular to the field, the contributions from the two edges perpendicular to the wire will cancel so we get

$$\oint \mathbf{B} \times d\mathbf{l} = \frac{\mu_0 I}{2\pi} \left(\frac{1}{a} - \frac{1}{a+1} \right) \quad (6)$$

where a is the distance of the near edge from the wire. This is clearly not zero, which it would have to be for the integral to be path independent.

Another attempt at a formula is

$$\mathbf{A}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) d\lambda \quad (7)$$

(This formula is stated in Griffiths's problem.)

For a constant field, this gives

$$\mathbf{A} = -\mathbf{r} \times \mathbf{B} \int_0^1 \lambda d\lambda \quad (8)$$

$$= -\frac{1}{2} \mathbf{r} \times \mathbf{B} \quad (9)$$

which is correct.

In the case of the infinite wire, we have

$$\mathbf{B}(\lambda \mathbf{r}) = \frac{\mu_0 I}{2\pi(\lambda s)} \hat{\phi} \quad (10)$$

where s is the perpendicular distance from the wire to the point \mathbf{r} . Therefore

$$\int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) d\lambda = \frac{\mu_0 I}{2\pi s} \hat{\phi} \int_0^1 d\lambda \quad (11)$$

$$= \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad (12)$$

In cylindrical coordinates, the position vector $\mathbf{r} = s\hat{\mathbf{s}} + z\hat{\mathbf{z}}$ so

$$\mathbf{A} = -\frac{\mu_0 I}{2\pi s} (s\hat{\mathbf{s}} + z\hat{\mathbf{z}}) \times \hat{\phi} \quad (13)$$

$$= \frac{\mu_0 I}{2\pi s} (z\hat{\mathbf{s}} - s\hat{\mathbf{z}}) \quad (14)$$

In this case, $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$ as can be checked by direct calculation, so this is a valid potential. However, in general, a potential calculated this way does not satisfy $\nabla \cdot \mathbf{A} = 0$. We have

$$\nabla \cdot \mathbf{A} = -\nabla \cdot \left[\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) d\lambda \right] \quad (15)$$

$$= \mathbf{r} \cdot \int_0^1 \lambda \nabla \times \mathbf{B}(\lambda \mathbf{r}) d\lambda - \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) \cdot \nabla \times \mathbf{r} d\lambda \quad (16)$$

$$= \mathbf{r} \cdot \int_0^1 \lambda^2 \mu_0 \mathbf{J}(\lambda \mathbf{r}) d\lambda - 0 \quad (17)$$

where we've used Ampère's law and the chain rule in the first term, and the fact that $\nabla \times \mathbf{r} = 0$ in the second. Thus the only way this can be zero for all observation points \mathbf{r} is for the integral to be zero, which effectively means that there can be no current. So this formula for the potential is not in general divergence-free.