

## MAGNETIC VECTOR POTENTIAL FROM MAGNETIC FIELD

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.51.

The magnetic vector potential can be used to find the field through the relation  $\mathbf{B} = \nabla \times \mathbf{A}$ , but there is no common formula for calculating the potential from the field. The potential is usually calculated from the current density via the formula

$$(0.1) \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

One attempt at a formula for deriving  $\mathbf{A}$  from  $\mathbf{B}$  is

$$(0.2) \quad \mathbf{A}(\mathbf{r}) = \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{B} \times d\mathbf{l}$$

where the integral is a line integral over some path from the origin (which must be defined independently) to the observation point  $\mathbf{r}$ . For a constant field, this gives

$$(0.3) \quad \mathbf{A} = \mathbf{B} \times \mathbf{r}$$

$$(0.4) \quad = -\mathbf{r} \times \mathbf{B}$$

This differs from the correct value by a factor of  $\frac{1}{2}$ , but there is a more serious problem in that the line integral isn't always independent of the path. To see this, consider the case of an infinite wire with a current  $I$ . The magnetic field circles the wire so in cylindrical coordinates

$$(0.5) \quad \mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

If we integrate  $\mathbf{B}$  around a closed square loop of side  $1$  that is perpendicular to the field, the contributions from the two edges perpendicular to the wire will cancel so we get

$$(0.6) \quad \oint \mathbf{B} \times d\mathbf{l} = \frac{\mu_0 I}{2\pi} \left( \frac{1}{a} - \frac{1}{a+1} \right)$$

where  $a$  is the distance of the near edge from the wire. This is clearly not zero, which it would have to be for the integral to be path independent.

Another attempt at a formula is

$$(0.7) \quad \mathbf{A}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) d\lambda$$

(This formula is stated in Griffiths's problem.)

For a constant field, this gives

$$(0.8) \quad \mathbf{A} = -\mathbf{r} \times \mathbf{B} \int_0^1 \lambda d\lambda$$

$$(0.9) \quad = -\frac{1}{2} \mathbf{r} \times \mathbf{B}$$

which is correct.

In the case of the infinite wire, we have

$$(0.10) \quad \mathbf{B}(\lambda \mathbf{r}) = \frac{\mu_0 I}{2\pi(\lambda s)} \hat{\phi}$$

where  $s$  is the perpendicular distance from the wire to the point  $\mathbf{r}$ . Therefore

$$(0.11) \quad \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) d\lambda = \frac{\mu_0 I}{2\pi s} \hat{\phi} \int_0^1 d\lambda$$

$$(0.12) \quad = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

In cylindrical coordinates, the position vector  $\mathbf{r} = s\hat{\mathbf{s}} + z\hat{\mathbf{z}}$  so

$$(0.13) \quad \mathbf{A} = -\frac{\mu_0 I}{2\pi s} (s\hat{\mathbf{s}} + z\hat{\mathbf{z}}) \times \hat{\phi}$$

$$(0.14) \quad = \frac{\mu_0 I}{2\pi s} (z\hat{\mathbf{s}} - s\hat{\mathbf{z}})$$

In this case,  $\nabla \cdot \mathbf{A} = 0$  and  $\nabla \times \mathbf{A} = \mathbf{B}$  as can be checked by direct calculation, so this is a valid potential. However, in general, a potential calculated this way does not satisfy  $\nabla \cdot \mathbf{A} = 0$ . We have

$$(0.15) \quad \nabla \cdot \mathbf{A} = -\nabla \cdot \left[ \mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) d\lambda \right]$$

$$(0.16) \quad = \mathbf{r} \cdot \int_0^1 \lambda \nabla \times \mathbf{B}(\lambda \mathbf{r}) d\lambda - \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r}) \cdot \nabla \times \mathbf{r} d\lambda$$

$$(0.17) \quad = \mathbf{r} \cdot \int_0^1 \lambda^2 \mu_0 \mathbf{J}(\lambda \mathbf{r}) d\lambda - 0$$

where we've used Ampère's law and the chain rule in the first term, and the fact that  $\nabla \times \mathbf{r} = 0$  in the second. Thus the only way this can be zero for all observation points  $\mathbf{r}$  is for the integral to be zero, which effectively means that there can be no current. So this formula for the potential is not in general divergence-free.