

## MAGNETIC FIELD: UNIQUENESS CONDITIONS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.54.

We've seen that solutions to Laplace's and Poisson's equations are unique, so it's natural to ask if a similar condition exists in the case of the magnetic field and its vector potential. That is, suppose we specify the field  $\mathbf{B}$  or the potential  $\mathbf{A}$  on the boundary of some volume, and also specify the current density  $\mathbf{J}$  within that volume. Does this determine the field uniquely within the volume?

First, we need an identity which can be derived from the divergence theorem and a vector calculus identity. For some vector fields  $\mathbf{U}$  and  $\mathbf{V}$ , we have

(0.1)

$$\int_A \mathbf{U} \times (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \int_V \nabla \cdot [\mathbf{U} \times (\nabla \times \mathbf{V})] d^3\mathbf{r}$$

(0.2)

$$= \int_V [(\nabla \times \mathbf{U}) \cdot (\nabla \times \mathbf{V}) - \mathbf{U} \cdot (\nabla \times (\nabla \times \mathbf{V}))] d^3\mathbf{r}$$

Following the same logic as in the electrostatic case, we suppose that there are two different solutions  $\mathbf{B}_1$  and  $\mathbf{B}_2$  (with corresponding potentials  $\mathbf{A}_1$  and  $\mathbf{A}_2$ ) and consider the difference  $\mathbf{B}_3 \equiv \mathbf{B}_2 - \mathbf{B}_1$ . Because the curl operator is linear, we have

(0.3)

$$\nabla \times \mathbf{A}_3 = \nabla \times (\mathbf{A}_2 - \mathbf{A}_1)$$

(0.4)

$$= \mathbf{B}_2 - \mathbf{B}_1$$

(0.5)

$$= \mathbf{B}_3$$

Setting  $\mathbf{U} = \mathbf{V} = \mathbf{A}_3$  in the identity above, we get on the LHS

(0.6)

$$\int_A \mathbf{U} \times (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \int_A (\mathbf{A}_3 \times \mathbf{B}_3) \cdot d\mathbf{a}$$

Note that if we specify either  $\mathbf{A}$  or  $\mathbf{B}$  on the boundary surface, then either  $\mathbf{A}_3 = 0$  or  $\mathbf{B}_3 = 0$  since there can be either a unique potential or a unique field on the boundary. Thus this surface integral is always zero.

It's important to note here that specifying  $\mathbf{A}$  on the boundary only does *not* specify  $\mathbf{B}$  on the boundary as well, since we need to know  $\mathbf{A}$  in three dimensions in order to calculate its curl.

On the RHS, we get

$$(0.7) \quad 0 = \int_V [\mathbf{B}_3 \cdot \mathbf{B}_3 - \mathbf{A}_3 \cdot \mathbf{J}_3] d^3 \mathbf{r}$$

Since we've specified the current inside the volume  $\mathbf{J}_3 = \mathbf{J}_2 - \mathbf{J}_1 = 0$  so we get

$$(0.8) \quad \int_V B_3^2 d^3 \mathbf{r} = 0$$

Since the integral of a positive definite quantity over the volume is zero, it must be zero everywhere, so  $\mathbf{B}_3 = 0$  and  $\mathbf{B}_2 = \mathbf{B}_1$ .