MAGNETIC FIELD: UNIQUENESS CONDITIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.54.

We've seen that solutions to Laplace's and Poisson's equations are unique, so it's natural to ask if a similar condition exists in the case of the magnetic field and its vector potential. That is, suppose we specify the field $\mathbf{B}$ or the potential $\mathbf{A}$ on the boundary of some volume, and also specify the current density $\mathbf{J}$ within that volume. Does this determine the field uniquely within the volume?

First, we need an identity which can be derived from the divergence theorem and a vector calculus identity. For some vector fields $\mathbf{U}$ and $\mathbf{V}$, we have

$$\int_A \mathbf{U}\times(\nabla\times\mathbf{V})\cdot d\mathbf{a} = \int_V \nabla\cdot[\mathbf{U}\times(\nabla\times\mathbf{V})]\,d^3\mathbf{r} \tag{1}$$

$$= \int_V [(\nabla\times\mathbf{U})\cdot(\nabla\times\mathbf{V}) - \mathbf{U}\cdot(\nabla\times(\nabla\times\mathbf{V}))]\,d^3\mathbf{r} \tag{2}$$

Following the same logic as in the electrostatic case, we suppose that there are two different solutions $\mathbf{B}_1$ and $\mathbf{B}_2$ (with corresponding potentials $\mathbf{A}_1$ and $\mathbf{A}_2$) and consider the difference $\mathbf{B}_3 \equiv \mathbf{B}_2 - \mathbf{B}_1$. Because the curl operator is linear, we have

$$\nabla\times\mathbf{A}_3 = \nabla\times(\mathbf{A}_2-\mathbf{A}_1) \tag{3}$$

$$= \mathbf{B}_2-\mathbf{B}_1 \tag{4}$$

$$= \mathbf{B}_3 \tag{5}$$

Setting $\mathbf{U}=\mathbf{V}=\mathbf{A}_3$ in the identity above, we get on the LHS

$$\int_A \mathbf{U}\times(\nabla\times\mathbf{V})\cdot d\mathbf{a} = \int_A (\mathbf{A}_3\times\mathbf{B}_3)\cdot d\mathbf{a} \tag{6}$$

Note that if we specify either $\mathbf{A}$ or $\mathbf{B}$ on the boundary surface, then either $\mathbf{A}_3 = 0$ or $\mathbf{B}_3 = 0$ since there can be either a unique potential or a unique field on the boundary. Thus this surface integral is always zero.

It's important to note here that specifying $\mathbf{A}$ on the boundary only does *not* specify $\mathbf{B}$ on the boundary as well, since we need to know $\mathbf{A}$ in three dimensions in order to calculate its curl.

On the RHS, we get

$$0=\int_V\left[\mathbf{B}_3\cdot\mathbf{B}_3-\mathbf{A}_3\cdot\mathbf{J}_3\right]d^3\mathbf{r} \tag{7}$$

Since we've specified the current inside the volume $\mathbf{J}_3=\mathbf{J}_2-\mathbf{J}_1=0$ so we get

$$\int_V B_3^2 d^3\mathbf{r}=0 \tag{8}$$

Since the integral of a positive definite quantity over the volume is zero, it must be zero everywhere, so $\mathbf{B}_3=0$ and $\mathbf{B}_2=\mathbf{B}_1$.