

MAGNETIC FIELD: UNIQUENESS CONDITIONS

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.54.

We've seen that solutions to Laplace's and Poisson's equations are unique, so it's natural to ask if a similar condition exists in the case of the magnetic field and its vector potential. That is, suppose we specify the field \mathbf{B} or the potential \mathbf{A} on the boundary of some volume, and also specify the current density \mathbf{J} within that volume. Does this determine the field uniquely within the volume?

First, we need an identity which can be derived from the divergence theorem and a vector calculus identity. For some vector fields \mathbf{U} and \mathbf{V} , we have

$$\begin{aligned} (1) \quad \int_A \mathbf{U} \times (\nabla \times \mathbf{V}) \cdot d\mathbf{a} &= \int_V \nabla \cdot [\mathbf{U} \times (\nabla \times \mathbf{V})] d^3\mathbf{r} \\ (2) \quad &= \int_V [(\nabla \times \mathbf{U}) \cdot (\nabla \times \mathbf{V}) - \mathbf{U} \cdot (\nabla \times (\nabla \times \mathbf{V}))] d^3\mathbf{r} \end{aligned}$$

Following the same logic as in the electrostatic case, we suppose that there are two different solutions \mathbf{B}_1 and \mathbf{B}_2 (with corresponding potentials \mathbf{A}_1 and \mathbf{A}_2) and consider the difference $\mathbf{B}_3 \equiv \mathbf{B}_2 - \mathbf{B}_1$. Because the curl operator is linear, we have

$$\begin{aligned} (3) \quad \nabla \times \mathbf{A}_3 &= \nabla \times (\mathbf{A}_2 - \mathbf{A}_1) \\ (4) \quad &= \mathbf{B}_2 - \mathbf{B}_1 \\ (5) \quad &= \mathbf{B}_3 \end{aligned}$$

Setting $\mathbf{U} = \mathbf{V} = \mathbf{A}_3$ in the identity above, we get on the LHS

$$(6) \quad \int_A \mathbf{U} \times (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \int_A (\mathbf{A}_3 \times \mathbf{B}_3) \cdot d\mathbf{a}$$

Note that if we specify either \mathbf{A} or \mathbf{B} on the boundary surface, then either $\mathbf{A}_3 = 0$ or $\mathbf{B}_3 = 0$ since there can be either a unique potential or a unique field on the boundary. Thus this surface integral is always zero.

It's important to note here that specifying \mathbf{A} on the boundary only does *not* specify \mathbf{B} on the boundary as well, since we need to know \mathbf{A} in three dimensions in order to calculate its curl.

On the RHS, we get

$$(7) \quad 0 = \int_V [\mathbf{B}_3 \cdot \mathbf{B}_3 - \mathbf{A}_3 \cdot \mathbf{J}_3] d^3 \mathbf{r}$$

Since we've specified the current inside the volume $\mathbf{J}_3 = \mathbf{J}_2 - \mathbf{J}_1 = 0$ so we get

$$(8) \quad \int_V B_3^2 d^3 \mathbf{r} = 0$$

Since the integral of a positive definite quantity over the volume is zero, it must be zero everywhere, so $\mathbf{B}_3 = 0$ and $\mathbf{B}_2 = \mathbf{B}_1$.