AVERAGE MAGNETIC FIELD WITHIN A SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.57.

We worked out the average electric field at the centre of a sphere earlier, so can we do something similar for the average magnetic field? Suppose we are given a steady current density within the sphere.

The starting point is the definition of the average field:

$$\mathbf{B}_{av}=\frac{3}{4\pi R^{3}}\int_{V}\mathbf{B}d^{3}\mathbf{r} \tag{1}$$

We can write this in terms of the vector potential and then in terms of the current density:

$$\begin{aligned}\frac{3}{4\pi R^{2}}\int_{V}\mathbf{B}d^{3}\mathbf{r} &= \frac{3}{4\pi R^{3}}\int_{V}\nabla\times\mathbf{A}d^{3}\mathbf{r} &(2)\\ &= -\frac{3}{4\pi R^{3}}\int_{A}\mathbf{A}\times d\mathbf{a} &(3)\\ &= -\frac{3\mu_{0}}{16\pi^{2}R^{3}}\int_{V}\int_{A}\frac{\mathbf{J}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\times d\mathbf{a}d^{3}\mathbf{r} &(4)\end{aligned}$$

In the second line we used the vector identity $\int_{V}\nabla\times\mathbf{A}d^{3}\mathbf{r}=-\int_{A}\mathbf{A}\times d\mathbf{a}$ and in line 3 we used the definition of the vector potential.

We are now faced with a surface integral *and* a volume integral. We can do the surface integral first, but we need to be clear about which coordinates are which. If we do the surface integral first, then we are choosing a particular volume element $d^{3}\mathbf{r}$ and holding it constant while we integrate over the surface. That is, the observation point $\mathbf{r}$ points to the volume element and the source point $\mathbf{r}'$ points to the surface element. This is why we've explicitly stated the dependence of $\mathbf{J}(\mathbf{r})$ since it depends on the volume element, and not the surface element.

The only term containing $\mathbf{r}'$ is therefore the $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ factor, so the integral we need to do is $\int_{A}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{a}$. Since $\mathbf{r}$ is fixed, we can point it along the z axis, making θ the angle between $\mathbf{r}$ and $\mathbf{r}'$. Also, since $\mathbf{r}'$ always points to a surface element, its magnitude is always R. We therefore have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{\sqrt{r^2+R^2-2rR\cos\theta}} \tag{5}$$

The surface element $d\mathbf{a}$ always points radially outward since it's on the surface of a sphere, so by symmetry the integral over the x and y components of $d\mathbf{a}$ will cancel out and we're left with only the z component, which is $\cos\theta\,|d\mathbf{a}|=R^2\sin\theta\cos\theta d\theta d\phi$. The integral we must do is thus

$$\int_A \frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{a}=R^2\hat{\mathbf{z}}\int_0^{\pi}\int_0^{2\pi}\frac{\sin\theta\cos\theta d\theta d\phi}{\sqrt{r^2+R^2-2rR\cos\theta}} \tag{6}$$

Using software, we find that

$$\int\frac{\sin\theta\cos\theta d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}}=\frac{1}{3}\frac{\sqrt{r^2+R^2-2rR\cos(\theta)}\left(r^2+R^2+rR\cos(\theta)\right)}{r^2} \tag{7}$$

Evaluating the limits requires specifying whether R is larger or smaller than r. If we're interested in currents within the sphere, then $R>r>0$ and we have

$$R^2\hat{\mathbf{z}}\int_0^{\pi}\int_0^{2\pi}\frac{\sin\theta\cos\theta d\theta d\phi}{\sqrt{r^2+R^2-2rR\cos\theta}}=\frac{4}{3}\pi r\hat{\mathbf{z}} \tag{8}$$

Plugging this back into the average field formula above we get

$$\mathbf{B}_{av}=-\frac{\mu_0}{4\pi R^3}\int_V \mathbf{J}\times\mathbf{r}d^3\mathbf{r} \tag{9}$$

We can write this in terms of the magnetic dipole moment as follows:

$$\mathbf{B}_{av}=\frac{2\mu_0}{4\pi R^3}\mathbf{m} \tag{10}$$

If we look at currents *outside* the sphere, then $R<r$ when we evaluate the limits on the integral above, giving

$$R^2\hat{\mathbf{z}}\int_0^{\pi}\int_0^{2\pi}\frac{\sin\theta\cos\theta d\theta d\phi}{\sqrt{r^2+R^2-2rR\cos\theta}}=\frac{2R^3}{3r^2}\hat{\mathbf{z}} \tag{11}$$

Plugging this into the average field formula we get

$$\mathbf{B}_{av}=-\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}\times\mathbf{r}}{r^3}d^3\mathbf{r} \tag{12}$$

Remember that $\mathbf{r}$ points *from* the centre to the volume element, so it is the negative of the vector $\mathbf{r}_c$ from the volume element to the centre. In this form we get

$$\mathbf{B}_{av}=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}\times\mathbf{r}_c}{r_c^3}d^3\mathbf{r} \tag{13}$$

This is just the volume form of the Biot-Savart law for calculating the field at the centre of the sphere thus for currents outside the sphere, the average of their field over the sphere is equal to the field produced at the centre of the sphere.

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