

MAGNETIC POTENTIAL AND FIELD OF A ROTATING SPHERE OF CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.58.

We return to the rotating solid sphere of charge. If the sphere has radius R , contains total charge Q and is rotating with angular speed ω we can find its magnetic moment from the gyromagnetic ratio. The angular momentum of a sphere is $L = \frac{2}{5}MR^2\omega$, so

$$(0.1) \quad \gamma = \frac{m}{L} = \frac{Q}{2M}$$

$$(0.2) \quad m = \frac{LQ}{2M}$$

$$(0.3) \quad = \frac{1}{5}QR^2\omega$$

We can align the sphere so its magnetic moment is along the z axis. From this we can find the average field within the sphere:

$$(0.4) \quad \mathbf{B}_{av} = \frac{2\mu_0}{4\pi R^3} \mathbf{m}$$

$$(0.5) \quad = \frac{\mu_0}{4\pi} \frac{2Q\omega}{5R} \hat{\mathbf{z}}$$

We worked out the exact field earlier and found

$$(0.6) \quad \mathbf{B} = \frac{\mu_0\omega Q}{4\pi R} \left[\left(1 - \frac{3}{5} \frac{r^2}{R^2}\right) \cos\theta \hat{\mathbf{r}} + \left(\frac{6}{5} \frac{r^2}{R^2} - 1\right) \sin\theta \hat{\theta} \right]$$

$$(0.7) \quad = \frac{\mu_0\omega Q}{4\pi R} \left[\hat{\mathbf{z}} - \frac{3}{5} \frac{r^2}{R^2} \cos\theta \hat{\mathbf{r}} + \frac{6}{5} \frac{r^2}{R^2} \sin\theta \hat{\theta} \right]$$

We can calculate the average field within the sphere from this formula by finding

$$(0.8) \quad \mathbf{B}_{av} = \frac{3}{4\pi R^3} \int_V \mathbf{B} d^3\mathbf{r}$$

The average field has three contributions. The $\hat{\mathbf{z}}$ contribution is constant, so is just

$$(0.9) \quad \mathbf{B}_{av;z} = \frac{\mu_0 \omega Q}{4\pi R} \hat{\mathbf{z}}$$

The $\hat{\mathbf{r}}$ contribution, by symmetry, has a non-zero component only in the z direction, and since the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ is θ , this contribution will be $\mathbf{B}_r = -\frac{\mu_0 \omega Q}{4\pi R} \frac{3}{5} \frac{r^2}{R^2} \cos^2 \theta \hat{\mathbf{z}}$. If we take the average of this, we have

$$(0.10)$$

$$(0.11) \quad \begin{aligned} \mathbf{B}_{av;r} &= \frac{3}{4\pi R^3} \int_V \mathbf{B}_r d^3 \mathbf{r} \\ &= -\hat{\mathbf{z}} \frac{\mu_0 \omega Q}{4\pi R^3} \frac{3}{5} \frac{3}{4\pi R^3} \int_0^R \int_0^\pi \int_0^{2\pi} (r^2 \cos^2 \theta) (r^2 \sin \theta) d\phi d\theta dr \end{aligned}$$

$$(0.12) \quad = -\frac{3}{25} \frac{\mu_0 \omega Q}{4\pi R} \hat{\mathbf{z}}$$

For the $\hat{\theta}$ contribution, again the non-zero component is in the z direction. This time, the angle between $\hat{\theta}$ and $\hat{\mathbf{z}}$ is $\theta + \frac{\pi}{2}$ and $\cos(\theta + \frac{\pi}{2}) = -\sin \theta$, so the contribution is $\mathbf{B}_\theta = -\frac{\mu_0 \omega Q}{4\pi R} \frac{6}{5} \frac{r^2}{R^2} \sin^2 \theta \hat{\mathbf{z}}$. The average is

$$(0.13)$$

$$(0.14) \quad \begin{aligned} \mathbf{B}_{av;\theta} &= \frac{3}{4\pi R^3} \int_V \mathbf{B}_\theta d^3 \mathbf{r} \\ &= -\hat{\mathbf{z}} \frac{\mu_0 \omega Q}{4\pi R^3} \frac{6}{5} \frac{3}{4\pi R^3} \int_0^R \int_0^\pi \int_0^{2\pi} (r^2 \sin^2 \theta) (r^2 \sin \theta) d\phi d\theta dr \end{aligned}$$

$$(0.15) \quad = -\frac{12}{25} \frac{\mu_0 \omega Q}{4\pi R} \hat{\mathbf{z}}$$

The total average field is then

$$(0.16) \quad \mathbf{B}_{av} = \mathbf{B}_{av;z} + \mathbf{B}_{av;r} + \mathbf{B}_{av;\theta}$$

$$(0.17) \quad = \left(1 - \frac{3}{25} - \frac{12}{25} \right) \frac{\mu_0 \omega Q}{4\pi R} \hat{\mathbf{z}}$$

$$(0.18) \quad = \frac{2}{5} \frac{\mu_0 \omega Q}{4\pi R} \hat{\mathbf{z}}$$

This agrees with the calculation above using the magnetic moment. The dipole term in the vector potential is

$$(0.19) \quad \mathbf{A}_1 = \frac{\mu_0}{4\pi r^2} \mathbf{m} \times \hat{\mathbf{r}}$$

$$(0.20) \quad = \frac{\mu_0}{4\pi r^2} \frac{R^2 Q \omega}{5} \sin \theta \hat{\phi}$$

so this is what we'd expect the potential to be for large distances from the sphere. However, we've seen that for a spherical shell, the potential outside the sphere is exactly equal to that of a perfect dipole. We might expect, therefore, that for a solid sphere, the dipole potential is also the exact potential. The potential for the shell of radius r' is

$$(0.21) \quad \mathbf{A} = \frac{\mu_0 r'^4 \omega \sigma \sin \theta}{3} \frac{\hat{\phi}}{r^2}$$

The surface charge density σ for an infinitesimal shell of thickness dr' within the solid sphere is

$$(0.22) \quad \sigma = \frac{3Q}{4\pi R^3} dr'$$

so the total potential is (remember we're holding the observation point constant, so r and θ are both constants):

$$(0.23) \quad \mathbf{A} = \frac{\mu_0 \omega}{3} \frac{3Q}{4\pi R^3} \frac{\sin \theta}{r^2} \hat{\phi} \int_0^R r'^4 dr'$$

$$(0.24) \quad = \frac{\mu_0}{4\pi r^2} \frac{R^2 Q \omega}{5} \sin \theta \hat{\phi}$$

Thus the exact potential is indeed equal to the dipole potential.

PINGBACKS

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