MAGNETIC POTENTIAL AND FIELD OF A ROTATING SPHERE OF CHARGE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.58.

We return to the rotating solid sphere of charge. If the sphere has radius R, contains total charge Q and is rotating with angular speed ω we can find its magnetic moment from the gyromagnetic ratio. The angular momentum of a sphere is $L=\frac{2}{5}MR^2\omega$, so

$$\gamma = \frac{m}{L}=\frac{Q}{2M} \tag{1}$$

$$m = \frac{LQ}{2M} \tag{2}$$

$$= \frac{1}{5}QR^2\omega \tag{3}$$

We can align the sphere so its magnetic moment is along the z axis. From this we can find the average field within the sphere:

$$\mathbf{B}_{av} = \frac{2\mu_0}{4\pi R^3}\mathbf{m} \tag{4}$$

$$= \frac{\mu_0}{4\pi}\frac{2Q\omega}{5R}\hat{\mathbf{z}} \tag{5}$$

We worked out the exact field earlier and found

$$\mathbf{B}=\frac{\mu_0\omega Q}{4\pi R}\left[\left(1-\frac{3}{5}\frac{r^2}{R^2}\right)\cos\theta\hat{\mathbf{r}}+\left(\frac{6}{5}\frac{r^2}{R^2}-1\right)\sin\theta\hat{\theta}\right] \tag{6}$$

$$=\frac{\mu_0\omega Q}{4\pi R}\left[\hat{\mathbf{z}}-\frac{3}{5}\frac{r^2}{R^2}\cos\theta\hat{\mathbf{r}}+\frac{6}{5}\frac{r^2}{R^2}\sin\theta\hat{\theta}\right] \tag{7}$$

We can calculate the average field within the sphere from this formula by finding

$$\mathbf{B}_{av}=\frac{3}{4\pi R^3}\int_V \mathbf{B}d^3\mathbf{r} \tag{8}$$

The average field has three contributions. The $\hat{\mathbf{z}}$ contribution is constant, so is just

$$\mathbf{B}_{av;z} = \frac{\mu_0 \omega Q}{4\pi R}\hat{\mathbf{z}} \tag{9}$$

The $\hat{\mathbf{r}}$ contribution, by symmetry, has a non-zero component only in the z direction, and since the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ is θ, this contribution will be $\mathbf{B}_r = -\frac{\mu_0 \omega Q}{4\pi R}\frac{3}{5}\frac{r^2}{R^2}\cos^2\theta\hat{\mathbf{z}}$. If we take the average of this, we have

$$\begin{aligned}
\mathbf{B}_{av;r} &= \frac{3}{4\pi R^3}\int_V \mathbf{B}_r d^3\mathbf{r} && (10)\\
&= -\hat{\mathbf{z}}\frac{\mu_0 \omega Q}{4\pi R^3}\frac{3}{5}\frac{3}{4\pi R^3}\int_0^R\int_0^\pi\int_0^{2\pi}\left(r^2\cos^2\theta\right)\left(r^2\sin\theta\right)d\phi d\theta dr && (11)\\
&= -\frac{3}{25}\frac{\mu_0 \omega Q}{4\pi R}\hat{\mathbf{z}} && (12)
\end{aligned}$$

For the $\hat{\theta}$ contribution, again the non-zero component is in the z direction. This time, the angle between $\hat{\theta}$ and $\hat{\mathbf{z}}$ is $\theta + \frac{\pi}{2}$ and $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$, so the contribution is $\mathbf{B}_\theta = -\frac{\mu_0 \omega Q}{4\pi R}\frac{6}{5}\frac{r^2}{R^2}\sin^2\theta\hat{\mathbf{z}}$ The average is

$$\begin{aligned}
\mathbf{B}_{av;\theta} &= \frac{3}{4\pi R^3}\int_V \mathbf{B}_\theta d^3\mathbf{r} && (13)\\
&= -\hat{\mathbf{z}}\frac{\mu_0 \omega Q}{4\pi R^3}\frac{6}{5}\frac{3}{4\pi R^3}\int_0^R\int_0^\pi\int_0^{2\pi}\left(r^2\sin^2\theta\right)\left(r^2\sin\theta\right)d\phi d\theta dr && (14)\\
&= -\frac{12}{25}\frac{\mu_0 \omega Q}{4\pi R}\hat{\mathbf{z}} && (15)
\end{aligned}$$

The total average field is then

$$\begin{aligned}
\mathbf{B}_{av} &= \mathbf{B}_{av;z} + \mathbf{B}_{av;r} + \mathbf{B}_{av;\theta} && (16)\\
&= \left(1 - \frac{3}{25} - \frac{12}{25}\right)\frac{\mu_0 \omega Q}{4\pi R}\hat{\mathbf{z}} && (17)\\
&= \frac{2}{5}\frac{\mu_0 \omega Q}{4\pi R}\hat{\mathbf{z}} && (18)
\end{aligned}$$

This agrees with the calculation above using the magnetic moment.

The dipole term in the vector potential is

$$\begin{aligned}
\mathbf{A}_1 &= \frac{\mu_0}{4\pi r^2}\mathbf{m}\times\hat{\mathbf{r}} && (19)\\
&= \frac{\mu_0}{4\pi r^2}\frac{R^2 Q\omega}{5}\sin\theta\hat{\phi} && (20)
\end{aligned}$$

so this is what we'd expect the potential to be for large distances from the sphere. However, we've seen that for a spherical shell, the potential outside the sphere is exactly equal to that of a perfect dipole. We might expect, therefore, that for a solid sphere, the dipole potential is also the exact potential. The potential for the shell of radius r' is

$$\mathbf{A} = \frac{\mu_0 r'^4 \omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} \tag{21}$$

The surface charge density σ for an infinitesimal shell of thickness dr' within the solid sphere is

$$\sigma = \frac{3Q}{4\pi R^3}dr' \tag{22}$$

so the total potential is (remember we're holding the observation point constant, so r and θ are both constants):

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0\omega}{3}\frac{3Q}{4\pi R^3}\frac{\sin\theta}{r^2}\hat{\phi}\int_0^R r'^4 dr' && (23)\\
&= \frac{\mu_0}{4\pi r^2}\frac{R^2 Q\omega}{5}\sin\theta\hat{\phi} && (24)
\end{aligned}$$

Thus the exact potential is indeed equal to the dipole potential.

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