

## MAGNETIC DIPOLE: AVERAGE FIELD OVER A SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.59.

The magnetic field of a perfect dipole pointing the  $\hat{\mathbf{z}}$  direction is

$$(1) \quad \mathbf{B}_{dip} = \frac{\mu_0 m}{4r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

We can average this over a sphere using the same logic as we did in the case of the solid rotating sphere. The average is, by symmetry, in the  $z$  direction, and the  $r$  component contributes  $\cos \theta \hat{\mathbf{z}}$  and the  $\theta$  component contributes  $-\sin \theta \hat{\mathbf{z}}$ , so we get

$$(2) \quad \mathbf{B}_{av} = \frac{\mu_0 m}{4} \hat{\mathbf{z}} \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{r^3} (\cos^2 \theta - \sin^2 \theta) (r^2 \sin \theta) d\phi d\theta dr$$

$$(3) \quad = \frac{\mu_0 m}{4} \hat{\mathbf{z}} \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{r} (\cos^2 \theta \sin \theta - \sin^3 \theta) d\phi d\theta dr$$

If we do the integral over  $\theta$  first, we find that this integral works out to zero. Clearly this doesn't agree with the result we obtained earlier for the average magnetic field inside a sphere due to currents within the sphere:

$$(4) \quad \mathbf{B}_{av} = \frac{2\mu_0}{4\pi R^3} \mathbf{m}$$

The problem is actually the same one that we encountered when dealing with the electric dipole. The integral over  $r$  blows up at  $r = 0$  (since  $\ln r$  is infinite there); If we integrate over any lower limit  $\epsilon > 0$  out to  $R$ , the problem goes away and the integral is a well-behaved zero. The solution to our current problem is the same as in the electric dipole case: introduce a delta function component to the field. That is we add in a term  $K\delta^3(\mathbf{r})$  for some constant  $K$  in order to make the average field come out to what we want. We must have

$$(5) \quad \mathbf{B}_{av} = \frac{3}{4\pi R^3} K \int_{\mathcal{V}} \delta^3(\mathbf{r}) d^3\mathbf{r}$$

$$(6) \quad = \frac{2\mu_0 m}{4\pi R^3}$$

$$(7) \quad K = \frac{2}{3} \mu_0 m$$

As in the electric case, it's not really quite correct to write

$$(8) \quad \mathbf{B}_{dip} = \frac{\mu_0 m}{4r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) + \frac{2}{3} \mu_0 m \delta^3(\mathbf{r}) \hat{\mathbf{z}}$$

$$(9) \quad = \frac{\mu_0}{4\pi r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}] + \frac{2}{3} \mu_0 m \delta^3(\mathbf{r}) \hat{\mathbf{z}}$$

since the first term doesn't apply at  $r = 0$  but it's a convenient notation as long as it's interpreted with caution.