MAGNETIC DIPOLE: AVERAGE FIELD OVER A SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.59.

The magnetic field of a perfect dipole pointing the $\hat{\mathbf{z}}$ direction is

$$\mathbf{B}_{dip} = \frac{\mu_0 m}{4r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) \tag{1}$$

We can average this over a sphere using the same logic as we did in the case of the solid rotating sphere. The average is, by symmetry, in the z direction, and the r component contributes $\cos\theta\hat{\mathbf{z}}$ and the θ component contributes $-\sin\theta\hat{\mathbf{z}}$, so we get

$$\begin{aligned}
\mathbf{B}_{av} &= \frac{\mu_0 m}{4}\hat{\mathbf{z}}\int_0^R\int_0^\pi\int_0^{2\pi}\frac{1}{r^3}\left(\cos^2\theta - \sin^2\theta\right)\left(r^2\sin\theta\right)d\phi d\theta dr &(2)\\
&= \frac{\mu_0 m}{4}\hat{\mathbf{z}}\int_0^R\int_0^\pi\int_0^{2\pi}\frac{1}{r}\left(\cos^2\theta\sin\theta - \sin^3\theta\right)d\phi d\theta dr &(3)
\end{aligned}$$

If we do the integral over θ first, we find that this integral works out to zero. Clearly this doesn't agree with the result we obtained earlier for the average magnetic field inside a sphere due to currents within the sphere:

$$\mathbf{B}_{av} = \frac{2\mu_0}{4\pi R^3}\mathbf{m} \tag{4}$$

The problem is actually the same one that we encountered when dealing with the electric dipole. The integral over r blows up at $r = 0$ (since $\ln r$ is infinite there); If we integrate over any lower limit $\varepsilon > 0$ out to R, the problem goes away and the integral is a well-behaved zero. The solution to our current problem is the same as in the electric dipole case: introduce a delta function component to the field. That is we add in a term $K\delta^3(\mathbf{r})$ for some constant K in order to make the average field come out to what we want. We must have

$$\mathbf{B}_{av} = \frac{3}{4\pi R^3} K \int_{\mathcal{V}} \delta^3(\mathbf{r})\, d^3\mathbf{r} \tag{5}$$

$$= \frac{2\mu_0 m}{4\pi R^3} \tag{6}$$

$$K = \frac{2}{3}\mu_0 m \tag{7}$$

As in the electric case, it's not really quite correct to write

$$\mathbf{B}_{dip} = \frac{\mu_0 m}{4r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) + \frac{2}{3}\mu_0 m \delta^3(\mathbf{r})\hat{\mathbf{z}} \tag{8}$$

$$= \frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] + \frac{2}{3}\mu_0 m \delta^3(\mathbf{r})\hat{\mathbf{z}} \tag{9}$$

since the first term doesn't apply at $r = 0$ but it's a convenient notation as long as it's interpreted with caution.