

MAGNETIC DIPOLE: AVERAGE FIELD OVER A SPHERE

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.59.

The magnetic field of a perfect dipole pointing the $\hat{\mathbf{z}}$ direction is

$$\mathbf{B}_{dip} = \frac{\mu_0 m}{4r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \quad (1)$$

We can average this over a sphere using the same logic as we did in the case of the solid rotating sphere. The average is, by symmetry, in the z direction, and the r component contributes $\cos \theta \hat{\mathbf{z}}$ and the θ component contributes $-\sin \theta \hat{\mathbf{z}}$, so we get

$$\mathbf{B}_{av} = \frac{\mu_0 m}{4} \hat{\mathbf{z}} \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{r^3} (\cos^2 \theta - \sin^2 \theta) (r^2 \sin \theta) d\phi d\theta dr \quad (2)$$

$$= \frac{\mu_0 m}{4} \hat{\mathbf{z}} \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{r} (\cos^2 \theta \sin \theta - \sin^3 \theta) d\phi d\theta dr \quad (3)$$

If we do the integral over θ first, we find that this integral works out to zero. Clearly this doesn't agree with the result we obtained earlier for the average magnetic field inside a sphere due to currents within the sphere:

$$\mathbf{B}_{av} = \frac{2\mu_0}{4\pi R^3} \mathbf{m} \quad (4)$$

The problem is actually the same one that we encountered when dealing with the electric dipole. The integral over r blows up at $r = 0$ (since $\ln r$ is infinite there); If we integrate over any lower limit $\epsilon > 0$ out to R , the problem goes away and the integral is a well-behaved zero. The solution to our current problem is the same as in the electric dipole case: introduce a delta function component to the field. That is we add in a term $K \delta^3(\mathbf{r})$ for some constant K in order to make the average field come out to what we want. We must have

$$\mathbf{B}_{av} = \frac{3}{4\pi R^3} K \int_{\mathcal{V}} \delta^3(\mathbf{r}) d^3\mathbf{r} \quad (5)$$

$$= \frac{2\mu_0 m}{4\pi R^3} \quad (6)$$

$$K = \frac{2}{3}\mu_0 m \quad (7)$$

As in the electric case, it's not really quite correct to write

$$\mathbf{B}_{dip} = \frac{\mu_0 m}{4r^3} (2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) + \frac{2}{3}\mu_0 m\delta^3(\mathbf{r})\hat{\mathbf{z}} \quad (8)$$

$$= \frac{\mu_0}{4\pi r^3} [3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \frac{2}{3}\mu_0 m\delta^3(\mathbf{r})\hat{\mathbf{z}} \quad (9)$$

since the first term doesn't apply at $r = 0$ but it's a convenient notation as long as it's interpreted with caution.