

## MAGNETIC DIPOLE MOMENT FOR VOLUME CURRENT

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.60.

The expansion of the magnetic vector potential was done originally for line currents, but a more general expression is possible for volume currents. We start with the relation between the potential and current:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (1)$$

We can expand the integrand in terms of Legendre polynomials in the same way as for the electric potential:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta'}} \quad (2)$$

$$= \frac{1}{r} \frac{1}{\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r} \cos \theta'}} \quad (3)$$

$$= \frac{1}{r} \sum_{n=0}^{\infty} P_n(\cos \theta') \left(\frac{r'}{r}\right)^n \quad (4)$$

Substituting this into the potential, we get

$$\mathbf{A} = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_V P_n(\cos \theta') r'^n \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' \quad (5)$$

$$= \frac{\mu_0}{4\pi} \left[ \frac{1}{r} \int_V \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' + \frac{1}{r^2} \int_V r' \cos \theta' \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' + \dots \right] \quad (6)$$

The first integral can be rewritten by observing

$$\nabla \cdot (x\mathbf{J}) = \nabla x \cdot \mathbf{J} + x \nabla \cdot \mathbf{J} \quad (7)$$

$$= \hat{\mathbf{x}} \cdot \mathbf{J} + x \nabla \cdot \mathbf{J} \quad (8)$$

$$= J_x + x \nabla \cdot \mathbf{J} \quad (9)$$

For steady currents,  $\nabla \cdot \mathbf{J} = 0$ , so for the  $x$  component of the integral we have

$$\int_V J_x d^3 \mathbf{r}' = \int_V \nabla \cdot (x \mathbf{J}) d^3 \mathbf{r}' \quad (10)$$

$$= \int_A x \mathbf{J} \cdot d\mathbf{a}' \quad (11)$$

If the current is confined to a finite volume, we can take the surface of integration outside this volume, making the surface integral zero. A similar argument applies to the other two components of  $\mathbf{J}$ , so in general the monopole term in the expansion is zero.

The dipole term is a bit trickier. We can start with a similar vector identity to the above, but this time with two coordinates. With  $x$  and  $y$  we get:

$$\nabla \cdot (xy \mathbf{J}) = \nabla(xy) \cdot \mathbf{J} + xy \nabla \cdot \mathbf{J} \quad (12)$$

$$= x \nabla y \cdot \mathbf{J} + y \nabla x \cdot \mathbf{J} + xy \nabla \cdot \mathbf{J} \quad (13)$$

$$= x \hat{\mathbf{y}} \cdot \mathbf{J} + y \hat{\mathbf{x}} \cdot \mathbf{J} + xy \nabla \cdot \mathbf{J} \quad (14)$$

$$= x J_y + y J_x + xy \nabla \cdot \mathbf{J} \quad (15)$$

Again, taking  $\nabla \cdot \mathbf{J} = 0$  and converting to a surface integral we get

$$\int_V \nabla \cdot (xy \mathbf{J}) d^3 \mathbf{r} = \int_V (x J_y + y J_x) d^3 \mathbf{r} \quad (16)$$

$$= \int_A xy \mathbf{J} \cdot d\mathbf{a} \quad (17)$$

$$= 0 \quad (18)$$

$$\int_V x J_y d^3 \mathbf{r} = - \int_V y J_x d^3 \mathbf{r} \quad (19)$$

Choosing the two coordinates to be the same gives

$$\int_V x J_x d^3 \mathbf{r} = 0 \quad (20)$$

Returning to the dipole integral, we have for the  $x$  component (using  $r' \cos \theta' = \frac{\mathbf{r}}{r} \cdot \mathbf{r}'$ ):

$$\frac{1}{r^2} \int_V r' \cos \theta' J_x d^3 \mathbf{r}' = \frac{\mathbf{r}}{r^3} \cdot \int_V \mathbf{r}' J_x d^3 \mathbf{r}' \quad (21)$$

$$= \frac{1}{r^3} \int_V (xx' + yy' + zz') J_x d^3 \mathbf{r}' \quad (22)$$

$$= \frac{1}{2r^3} \int_V [y(y'J_x - x'J_y) + z(z'J_x - x'J_z)] d^3 \mathbf{r}' \quad (23)$$

$$= \frac{1}{2r^3} \int_V [y(\mathbf{J} \times \mathbf{r}')_z - z(\mathbf{J} \times \mathbf{r}')_y] d^3 \mathbf{r}' \quad (24)$$

$$= \frac{1}{2r^3} \left[ \mathbf{r} \times \int_V \mathbf{J} \times \mathbf{r}' d^3 \mathbf{r}' \right]_x \quad (25)$$

The other two components work out the same way. We can make the dipole term equivalent to the formula for line currents:

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi r^2} \mathbf{m} \times \hat{\mathbf{r}} \quad (26)$$

if we define the magnetic dipole moment here as

$$\mathbf{m} = -\frac{1}{2} \int_V \mathbf{J} \times \mathbf{r}' d^3 \mathbf{r}' \quad (27)$$

$$= \frac{1}{2} \int_V \mathbf{r}' \times \mathbf{J} d^3 \mathbf{r}' \quad (28)$$

PINGBACKS

Pingback: Average magnetic field within a sphere