

MAGNETIC DIPOLE MOMENT FOR VOLUME CURRENT

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.60.

The expansion of the magnetic vector potential was done originally for line currents, but a more general expression is possible for volume currents. We start with the relation between the potential and current:

$$(0.1) \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

We can expand the integrand in terms of Legendre polynomials in the same way as for the electric potential:

$$(0.2) \quad \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta'}}$$

$$(0.3) \quad = \frac{1}{r} \frac{1}{\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r} \cos \theta'}}$$

$$(0.4) \quad = \frac{1}{r} \sum_{n=0}^{\infty} P_n(\cos \theta') \left(\frac{r'}{r}\right)^n$$

Substituting this into the potential, we get

$$(0.5) \quad \mathbf{A} = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_V P_n(\cos \theta') r'^n \mathbf{J}(\mathbf{r}') d^3\mathbf{r}'$$

$$(0.6) \quad = \frac{\mu_0}{4\pi} \left[\frac{1}{r} \int_V \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' + \frac{1}{r^2} \int_V r' \cos \theta' \mathbf{J}(\mathbf{r}') d^3\mathbf{r}' + \dots \right]$$

The first integral can be rewritten by observing

$$(0.7) \quad \nabla \cdot (x\mathbf{J}) = \nabla x \cdot \mathbf{J} + x \nabla \cdot \mathbf{J}$$

$$(0.8) \quad = \hat{\mathbf{x}} \cdot \mathbf{J} + x \nabla \cdot \mathbf{J}$$

$$(0.9) \quad = J_x + x \nabla \cdot \mathbf{J}$$

For steady currents, $\nabla \cdot \mathbf{J} = 0$, so for the x component of the integral we have

$$(0.10) \quad \int_V J_x d^3 \mathbf{r}' = \int_V \nabla \cdot (x \mathbf{J}) d^3 \mathbf{r}'$$

$$(0.11) \quad = \int_A x \mathbf{J} \cdot d\mathbf{a}'$$

If the current is confined to a finite volume, we can take the surface of integration outside this volume, making the surface integral zero. A similar argument applies to the other two components of \mathbf{J} , so in general the monopole term in the expansion is zero.

The dipole term is a bit trickier. We can start with a similar vector identity to the above, but this time with two coordinates. With x and y we get:

$$(0.12) \quad \nabla \cdot (xy \mathbf{J}) = \nabla(xy) \cdot \mathbf{J} + xy \nabla \cdot \mathbf{J}$$

$$(0.13) \quad = x \nabla y \cdot \mathbf{J} + y \nabla x \cdot \mathbf{J} + xy \nabla \cdot \mathbf{J}$$

$$(0.14) \quad = x \hat{\mathbf{y}} \cdot \mathbf{J} + y \hat{\mathbf{x}} \cdot \mathbf{J} + xy \nabla \cdot \mathbf{J}$$

$$(0.15) \quad = x J_y + y J_x + xy \nabla \cdot \mathbf{J}$$

Again, taking $\nabla \cdot \mathbf{J} = 0$ and converting to a surface integral we get

$$(0.16) \quad \int_V \nabla \cdot (xy \mathbf{J}) d^3 \mathbf{r} = \int_V (x J_y + y J_x) d^3 \mathbf{r}$$

$$(0.17) \quad = \int_A xy \mathbf{J} \cdot d\mathbf{a}$$

$$(0.18) \quad = 0$$

$$(0.19) \quad \int_V x J_y d^3 \mathbf{r} = - \int_V y J_x d^3 \mathbf{r}$$

Choosing the two coordinates to be the same gives

$$(0.20) \quad \int_V x J_x d^3 \mathbf{r} = 0$$

Returning to the dipole integral, we have for the x component (using $r' \cos \theta' = \frac{r}{r'} \cdot \mathbf{r}'$):

$$(0.21) \quad \frac{1}{r^2} \int_V r' \cos \theta' J_x d^3 \mathbf{r}' = \frac{\mathbf{r}}{r^3} \cdot \int_V \mathbf{r}' J_x d^3 \mathbf{r}'$$

$$(0.22) \quad = \frac{1}{r^3} \int_V (xx' + yy' + zz') J_x d^3 \mathbf{r}'$$

$$(0.23) \quad = \frac{1}{2r^3} \int_V [y(y'J_x - x'J_y) + z(z'J_x - x'J_z)] d^3 \mathbf{r}'$$

$$(0.24) \quad = \frac{1}{2r^3} \int_V [y(\mathbf{J} \times \mathbf{r}')_z - z(\mathbf{J} \times \mathbf{r}')_y] d^3 \mathbf{r}'$$

$$(0.25) \quad = \frac{1}{2r^3} \left[\mathbf{r} \times \int_V \mathbf{J} \times \mathbf{r}' d^3 \mathbf{r}' \right]_x$$

The other two components work out the same way. We can make the dipole term equivalent to the formula for line currents:

$$(0.26) \quad \mathbf{A}_1 = \frac{\mu_0}{4\pi r^2} \mathbf{m} \times \hat{\mathbf{r}}$$

if we define the magnetic dipole moment here as

$$(0.27) \quad \mathbf{m} = -\frac{1}{2} \int_V \mathbf{J} \times \mathbf{r}' d^3 \mathbf{r}'$$

$$(0.28) \quad = \frac{1}{2} \int_V \mathbf{r}' \times \mathbf{J} d^3 \mathbf{r}'$$

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