MAGNETIC DIPOLE MOMENT FOR VOLUME CURRENT

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.60.

The expansion of the magnetic vector potential was done originally for line currents, but a more general expression is possible for volume currents. We start with the relation between the potential and current:

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{1}$$

We can expand the integrand in terms of Legendre polynomials in the same way as for the electric potential:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} \tag{2}$$

$$= \frac{1}{r}\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta'}} \tag{3}$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}P_n(\cos\theta')\left(\frac{r'}{r}\right)^n \tag{4}$$

Substituting this into the potential, we get

$$\mathbf{A}=\frac{\mu_0}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_V P_n(\cos\theta')r'^n\mathbf{J}(\mathbf{r}')\,d^3\mathbf{r}' \tag{5}$$

$$=\frac{\mu_0}{4\pi}\left[\frac{1}{r}\int_V\mathbf{J}(\mathbf{r}')\,d^3\mathbf{r}'+\frac{1}{r^2}\int_V r'\cos\theta'\mathbf{J}(\mathbf{r}')\,d^3\mathbf{r}'+\ldots\right] \tag{6}$$

The first integral can be rewritten by observing

$$\nabla\cdot(x\mathbf{J}) = \nabla x\cdot\mathbf{J}+x\nabla\cdot\mathbf{J} \tag{7}$$

$$= \hat{\mathbf{x}}\cdot\mathbf{J}+x\nabla\cdot\mathbf{J} \tag{8}$$

$$= J_x+x\nabla\cdot\mathbf{J} \tag{9}$$

For steady currents, $\nabla\cdot\mathbf{J}=0$, so for the x component of the integral we have

$$\int_V J_x d^3\mathbf{r}' = \int_V \nabla\cdot(x\mathbf{J})\,d^3\mathbf{r}' \tag{10}$$

$$= \int_A x\mathbf{J}\cdot d\mathbf{a}' \tag{11}$$

If the current is confined to a finite volume, we can take the surface of integration outside this volume, making the surface integral zero. A similar argument applies to the other two components of $\mathbf{J}$, so in general the monopole term in the expansion is zero.

The dipole term is a bit trickier. We can start with a similar vector identity to the above, but this time with two coordinates. With x and y we get:

$$\nabla\cdot(xy\mathbf{J}) = \nabla(xy)\cdot\mathbf{J}+xy\nabla\cdot\mathbf{J} \tag{12}$$

$$= x\nabla y\cdot\mathbf{J}+y\nabla x\cdot\mathbf{J}+xy\nabla\cdot\mathbf{J} \tag{13}$$

$$= x\hat{\mathbf{y}}\cdot\mathbf{J}+y\hat{\mathbf{x}}\cdot\mathbf{J}+xy\nabla\cdot\mathbf{J} \tag{14}$$

$$= xJ_y+yJ_x+xy\nabla\cdot\mathbf{J} \tag{15}$$

Again, taking $\nabla\cdot\mathbf{J}=0$ and converting to a surface integral we get

$$\int_V \nabla\cdot(xy\mathbf{J})\,d^3\mathbf{r} = \int_V (xJ_y+yJ_x)\,d^3\mathbf{r} \tag{16}$$

$$= \int_A xy\mathbf{J}\cdot d\mathbf{a} \tag{17}$$

$$= 0 \tag{18}$$

$$\int_V xJ_y d^3\mathbf{r} = -\int_v yJ_x d^3\mathbf{r} \tag{19}$$

Choosing the two coordinates to be the same gives

$$\int_V xJ_x d^3\mathbf{r} = 0 \tag{20}$$

Returning to the dipole integral, we have for the x component (using $r'\cos\theta' = \frac{\mathbf{r}}{r}\cdot\mathbf{r}'$):

$$\frac{1}{r^2}\int_V r'\cos\theta' J_x d^3\mathbf{r}' = \frac{\mathbf{r}}{r^3}\cdot\int_V \mathbf{r}' J_x d^3\mathbf{r}' \tag{21}$$

$$= \frac{1}{r^3}\int_V \left(xx'+yy'+zz'\right) J_x d^3\mathbf{r}' \tag{22}$$

$$= \frac{1}{2r^3}\int_V \left[y\left(y'J_x - x'J_y\right) + z\left(z'J_x - x'J_z\right)\right] d^3\mathbf{r}' \tag{23}$$

$$= \frac{1}{2r^3}\int_V \left[y\left(\mathbf{J}\times\mathbf{r}'\right)_z - z\left(\mathbf{J}\times\mathbf{r}'\right)_y\right] d^3\mathbf{r}' \tag{24}$$

$$= \frac{1}{2r^3}\left[\mathbf{r}\times\int_V \mathbf{J}\times\mathbf{r}' d^3\mathbf{r}'\right]_x \tag{25}$$

The other two components work out the same way. We can make the dipole term equivalent to the formula for line currents:

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi r^2}\mathbf{m}\times\hat{\mathbf{r}} \tag{26}$$

if we define the magnetic dipole moment here as

$$\mathbf{m} = -\frac{1}{2}\int_V \mathbf{J}\times\mathbf{r}' d^3\mathbf{r}' \tag{27}$$

$$= \frac{1}{2}\int_V \mathbf{r}'\times\mathbf{J} d^3\mathbf{r}' \tag{28}$$

Pingbacks

Pingback: Average magnetic field within a sphere