

## MAGNETIC DIPOLE FIELD OF A FINITE SOLENOID

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.61.

For our last post on magnetostatics, we'll consider a finite solenoid of radius  $R$  and length  $L$ , with a surface charge density  $\sigma$  rotating at angular speed  $\omega$ . We know that the field outside an infinite solenoid is zero, but what about a finite solenoid? For points far from the axis, we can use a dipole approximation.

We align the axis along the  $z$  axis, and consider it to be a stack of individual current loops, each with its own dipole moment. The moment of a current loop is

$$\mathbf{m} = \pi IR^2 \hat{\mathbf{z}} \quad (1)$$

In terms of the parameters of the problem, each loop has a thickness of  $dz$  and thus carries a current of  $I = \sigma R \omega dz$ . The contribution of the loop at coordinate  $z$  is therefore

$$d\mathbf{m} = \pi \omega \sigma R^3 dz \hat{\mathbf{z}} \quad (2)$$

The field of the dipole from this current loop is

$$d\mathbf{B}_{dip} = \frac{\mu_0}{4\pi r^3} [3(d\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - d\mathbf{m}] \quad (3)$$

We now need to consider carefully what is meant by  $\hat{\mathbf{r}}$ . We'll take the observation point to be a distance  $s$  along a line perpendicular to the axis and intersecting the axis at its midpoint. We'll define  $s$  to be on the  $\hat{\mathbf{x}}$  axis. Then for a given current loop at coordinate  $z$ , the vector  $\mathbf{r}$  points from the centre of this loop to  $s$ . Therefore

$$r = \sqrt{s^2 + z^2} \quad (4)$$

$$\cos \theta = -\frac{z}{r} \quad (5)$$

$$\sin \theta = \frac{s}{r} \quad (6)$$

$$d\mathbf{m} \cdot \hat{\mathbf{r}} = \pi \omega \sigma R^3 \cos \theta dz \quad (7)$$

$$= -\pi \omega \sigma R^3 \frac{z}{r} dz \quad (8)$$

where  $\theta$  is, as usual, the angle between  $\hat{\mathbf{r}}$  and  $\hat{\mathbf{z}}$ . Therefore

$$\hat{\mathbf{r}} = \cos \theta \hat{\mathbf{z}} + \sin \theta \hat{\mathbf{x}} \quad (9)$$

$$= -\frac{z}{r} \hat{\mathbf{z}} + \frac{s}{r} \hat{\mathbf{x}} \quad (10)$$

The total dipole field of the solenoid is therefore

$$\mathbf{B}_{dip} = \frac{\mu_0}{4\pi} \pi \omega \sigma R^3 \int_{-\frac{L}{2}}^{\frac{L}{2}} \left[ \left( \frac{3z^2}{(s^2 + z^2)^{5/2}} - \frac{1}{(s^2 + z^2)^{3/2}} \right) \hat{\mathbf{z}} + \frac{sz}{(s^2 + z^2)^{5/2}} \hat{\mathbf{x}} \right] dz \quad (11)$$

$$= -\frac{\mu_0 \omega \sigma R^3 L}{4 \left( s^2 + \left( \frac{L}{2} \right)^2 \right)^{3/2}} \hat{\mathbf{z}} + 0 \hat{\mathbf{x}} \quad (12)$$

$$= -\frac{\mu_0 \omega \sigma R^3 L}{4 \left( s^2 + \left( \frac{L}{2} \right)^2 \right)^{3/2}} \hat{\mathbf{z}} \quad (13)$$

Note that as  $L \rightarrow \infty$ , the field does tend to zero as  $\frac{1}{L^2}$  which is the correct value for an infinite solenoid.