MAGNETIC DIPOLE FIELD OF A FINITE SOLENOID

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.61.

For our last post on magnetostatics, we'll consider a finite solenoid of radius R and length L, with a surface charge density σ rotating at angular speed ω. We know that the field outside an infinite solenoid is zero, but what about a finite solenoid? For points far from the axis, we can use a dipole approximation.

We align the axis along the z axis, and consider it to be a stack of individual current loops, each with its own dipole moment. The moment of a current loop is

$$\mathbf{m} = \pi I R^2 \hat{\mathbf{z}} \tag{1}$$

In terms of the parameters of the problem, each loop has a thickness of dz and thus carries a current of $I = \sigma R \omega dz$. The contribution of the loop at coordinate z is therefore

$$d\mathbf{m} = \pi\omega\sigma R^3 dz \hat{\mathbf{z}} \tag{2}$$

The field of the dipole from this current loop is

$$d\mathbf{B}_{dip} = \frac{\mu_0}{4\pi r^3}\left[3\left(d\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - d\mathbf{m}\right] \tag{3}$$

We now need to consider carefully what is meant by $\hat{\mathbf{r}}$. We'll take the observation point to be a distance s along a line perpendicular to the axis and intersecting the axis at its midpoint. We'll define s to be on the $\hat{\mathbf{x}}$ axis. Then for a given current loop at coordinate z, the vector $\mathbf{r}$ points from the centre of this loop to s. Therefore

$$\begin{aligned}
r &= \sqrt{s^2+z^2} && (4)\\
\cos\theta &= -\frac{z}{r} && (5)\\
\sin\theta &= \frac{s}{r} && (6)\\
d\mathbf{m}\cdot\hat{\mathbf{r}} &= \pi\omega\sigma R^3 \cos\theta dz && (7)\\
&= -\pi\omega\sigma R^3 \frac{z}{r} dz && (8)
\end{aligned}$$

where θ is, as usual, the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$. Therefore

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\hat{\mathbf{z}}+\sin\theta\hat{\mathbf{x}} && (9)\\ &= -\frac{z}{r}\hat{\mathbf{z}}+\frac{s}{r}\hat{\mathbf{x}} && (10) \end{aligned}$$

The total dipole field of the solenoid is therefore

$$\mathbf{B}_{dip} = \frac{\mu_0}{4\pi}\pi\omega\sigma R^3\int_{-\frac{L}{2}}^{\frac{L}{2}}\left[\left(\frac{3z^2}{(s^2+z^2)^{5/2}}-\frac{1}{(s^2+z^2)^{3/2}}\right)\hat{\mathbf{z}}+\frac{sz}{(s^2+z^2)^{5/2}}\hat{\mathbf{x}}\right]dz \quad (11)$$

$$= -\frac{\mu_0\omega\sigma R^3 L}{4\left(s^2+\left(\frac{L}{2}\right)^2\right)^{3/2}}\hat{\mathbf{z}}+0\hat{\mathbf{x}} \quad (12)$$

$$= -\frac{\mu_0\omega\sigma R^3 L}{4\left(s^2+\left(\frac{L}{2}\right)^2\right)^{3/2}}\hat{\mathbf{z}} \quad (13)$$

Note that as $L\to\infty$, the field does tend to zero as $\frac{1}{L^2}$ which is the correct value for an infinite solenoid.