

TORQUE ON A MAGNETIC MOMENT; GENERAL CURRENT DISTRIBUTION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.2.

Jackson, J. D. (1999) Classical Electrodynamics, 3rd Edition; Wiley - Sections 5.6 and 5.7.

The torque on a square current loop in a constant, uniform magnetic field is given by

$$\mathbf{N} = \mathbf{m} \times \mathbf{B} \quad (1)$$

It turns out that this formula applies to any current distribution, although the proof is a bit tricky. Since the magnetic moment for a collection of line currents is defined as

$$\mathbf{m} = I\mathbf{a} = \frac{1}{2}I \int \mathbf{r} \times d\mathbf{l} \quad (2)$$

the natural generalization of magnetic moment to a general volume current density is

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{J} d^3\mathbf{r} \quad (3)$$

The Lorentz force law for a volume current is

$$\mathbf{F} = \int \mathbf{J} \times \mathbf{B} d^3\mathbf{r} \quad (4)$$

so the torque is

$$\mathbf{N} = \int \mathbf{r} \times (\mathbf{J} \times \mathbf{B}) d^3\mathbf{r} \quad (5)$$

We can convert this using a vector identity:

$$\mathbf{N} = \int (\mathbf{r} \cdot \mathbf{B}) \mathbf{J} d^3\mathbf{r} - \int (\mathbf{r} \cdot \mathbf{J}) \mathbf{B} d^3\mathbf{r} \quad (6)$$

To proceed further, we need another vector identity, which states that for scalar fields f and g and localized vector field \mathbf{J} , we have

$$\int [f\mathbf{J} \cdot \nabla g + g\mathbf{J} \cdot \nabla f + fg\nabla \cdot \mathbf{J}] d^3\mathbf{r} = 0 \quad (7)$$

This follows, since if we integrate the middle term by parts, we get

$$\int g\mathbf{J} \cdot \nabla f = fg(J_x + J_y + J_z) - \int f\nabla \cdot (g\mathbf{J}) d^3\mathbf{r} \quad (8)$$

$$= 0 - \int [f\mathbf{J} \cdot \nabla g + fg\nabla \cdot \mathbf{J}] d^3\mathbf{r} \quad (9)$$

where the integrated term is zero if we assume that \mathbf{J} goes to zero at infinity (that is, it's localized).

Now if we consider steady currents, then $\nabla \cdot \mathbf{J} = 0$ so the identity reduces to

$$\int [f\mathbf{J} \cdot \nabla g + g\mathbf{J} \cdot \nabla f] d^3\mathbf{r} = 0 \quad (10)$$

If we take $f = g = r = \sqrt{x^2 + y^2 + z^2}$, then $\nabla r = \mathbf{r}/r$ and

$$2 \int r\mathbf{J} \cdot \frac{\mathbf{r}}{r} d^3\mathbf{r} = 2 \int \mathbf{r} \cdot \mathbf{J} d^3\mathbf{r} = 0 \quad (11)$$

so the second integral in the torque 6 is zero (remember \mathbf{B} is constant, so it comes outside the integral).

For the first term, we use the identity with $f = x$, $g = y$:

$$\int (xJ_y + yJ_x) d^3\mathbf{r} = 0 \quad (12)$$

Choosing the other coordinates in turn, we have in general for i and j equal to any combination of x , y and z :

$$\int (r_i J_j + r_j J_i) d^3\mathbf{r} = 0 \quad (13)$$

Now look at the i th component of the first term of 6:

$$N_i = \int (\mathbf{r} \cdot \mathbf{B}) J_i d^3\mathbf{r} \quad (14)$$

$$= \sum_j B_j \int r_j J_i d^3\mathbf{r} \quad (15)$$

$$= \frac{1}{2} \sum_j B_j \int (r_j J_i - r_i J_j) d^3\mathbf{r} \quad (16)$$

where we used 13 to get the last line.

If we take $i = x$ for example, we get

$$N_x = \frac{1}{2}B_y \int (yJ_x - xJ_y) d^3\mathbf{r} + \frac{1}{2}B_z \int (zJ_x - xJ_z) d^3\mathbf{r} \quad (17)$$

$$= \frac{1}{2} \int \left[-(\mathbf{r} \times \mathbf{J})_z B_y + (\mathbf{r} \times \mathbf{J})_y B_z \right] d^3\mathbf{r} \quad (18)$$

$$= \frac{1}{2} \left[\left(\int \mathbf{r} \times \mathbf{J} d^3\mathbf{r} \right) \times \mathbf{B} \right]_x \quad (19)$$

The other two components work out similarly, so we get

$$\mathbf{N} = \left[\frac{1}{2} \int \mathbf{r} \times \mathbf{J} d^3\mathbf{r} \right] \times \mathbf{B} \quad (20)$$

$$= \mathbf{m} \times \mathbf{B} \quad (21)$$

I suspect this isn't the way Griffiths meant this problem to be solved, since he deals only with line currents, but this is a more general proof anyway.