TORQUE ON A MAGNETIC MOMENT; GENERAL CURRENT DISTRIBUTION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.2.

Jackson, J. D. (1999) Classical Electrodynamics, 3rd Edition; Wiley - Sections 5.6 and 5.7.

The torque on a square current loop in a constant, uniform magnetic field is given by

$$\mathbf{N}=\mathbf{m}\times\mathbf{B} \tag{1}$$

It turns out that this formula applies to any current distribution, although the proof is a bit tricky. Since the magnetic moment for a collection of line currents is defined as

$$\mathbf{m}=I\mathbf{a}=\frac{1}{2}I\int\mathbf{r}\times d\mathbf{l} \tag{2}$$

the natural generalization of magnetic moment to a general volume current density is

$$\mathbf{m}=\frac{1}{2}\int\mathbf{r}\times\mathbf{J}d^{3}\mathbf{r} \tag{3}$$

The Lorentz force law for a volume current is

$$\mathbf{F}=\int\mathbf{J}\times\mathbf{B}d^{3}\mathbf{r} \tag{4}$$

so the torque is

$$\mathbf{N}=\int\mathbf{r}\times(\mathbf{J}\times\mathbf{B})\,d^{3}\mathbf{r} \tag{5}$$

We can convert this using a vector identity:

$$\mathbf{N}=\int(\mathbf{r}\cdot\mathbf{B})\,\mathbf{J}d^{3}\mathbf{r}-\int(\mathbf{r}\cdot\mathbf{J})\,\mathbf{B}d^{3}\mathbf{r} \tag{6}$$

To proceed further, we need another vector identity, which states that for scalar fields f and g and localized vector field $\mathbf{J}$, we have

$$\int\left[f\mathbf{J}\cdot\nabla g+g\mathbf{J}\cdot\nabla f+fg\nabla\cdot\mathbf{J}\right]d^{3}\mathbf{r}=0 \tag{7}$$

This follows, since if we integrate the middle term by parts, we get

$$\int g\mathbf{J}\cdot\nabla f=fg\left(J_{x}+J_{y}+J_{z}\right)-\int f\nabla\cdot\left(g\mathbf{J}\right)d^{3}\mathbf{r} \tag{8}$$

$$=0-\int\left[f\mathbf{J}\cdot\nabla g+fg\nabla\cdot\mathbf{J}\right]d^{3}\mathbf{r} \tag{9}$$

where the integrated term is zero if we assume that $\mathbf{J}$ goes to zero at infinity (that is, it's localized).

Now if we consider steady currents, then $\nabla\cdot\mathbf{J}=0$ so the identity reduces to

$$\int\left[f\mathbf{J}\cdot\nabla g+g\mathbf{J}\cdot\nabla f\right]d^{3}\mathbf{r}=0 \tag{10}$$

If we take $f=g=r=\sqrt{x^{2}+y^{2}+z^{2}}$, then $\nabla r=\mathbf{r}/r$ and

$$2\int r\mathbf{J}\cdot\frac{\mathbf{r}}{r}d^{3}\mathbf{r}=2\int\mathbf{r}\cdot\mathbf{J}d^{3}\mathbf{r}=0 \tag{11}$$

so the second integral in the torque 6 is zero (remember $\mathbf{B}$ is constant, so it comes outside the integral).

For the first term, we use the identity with $f=x$, $g=y$:

$$\int\left(xJ_{y}+yJ_{x}\right)d^{3}\mathbf{r}=0 \tag{12}$$

Choosing the other coordinates in turn, we have in general for i and j equal to any combination of x, y and z:

$$\int\left(r_{i}J_{j}+r_{j}J_{i}\right)d^{3}\mathbf{r}=0 \tag{13}$$

Now look at the ith component of the first term of 6:

$$N_i = \int (\mathbf{r}\cdot\mathbf{B}) J_i d^3\mathbf{r} \tag{14}$$

$$= \sum_j B_j \int r_j J_i d^3\mathbf{r} \tag{15}$$

$$= \frac{1}{2}\sum_j B_j \int \left(r_j J_i - r_i J_j\right) d^3\mathbf{r} \tag{16}$$

where we used 13 to get the last line.

If we take $i = x$ for example, we get

$$N_x = \frac{1}{2}B_y \int (yJ_x - xJ_y)\, d^3\mathbf{r} + \frac{1}{2}B_z \int (zJ_x - xJ_z)\, d^3\mathbf{r} \tag{17}$$

$$= \frac{1}{2}\int \left[-(\mathbf{r}\times\mathbf{J})_z B_y + (\mathbf{r}\times\mathbf{J})_y B_z\right] d^3\mathbf{r} \tag{18}$$

$$= \frac{1}{2}\left[\left(\int \mathbf{r}\times\mathbf{J} d^3\mathbf{r}\right)\times\mathbf{B}\right]_x \tag{19}$$

The other two components work out similarly, so we get

$$\mathbf{N} = \left[\frac{1}{2}\int \mathbf{r}\times\mathbf{J} d^3\mathbf{r}\right]\times\mathbf{B} \tag{20}$$

$$= \mathbf{m}\times\mathbf{B} \tag{21}$$

I suspect this isn't the way Griffiths meant this problem to be solved, since he deals only with line currents, but this is a more general proof anyway.