FORCE BETWEEN TWO MAGNETIC DIPOLES

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.3.

A general formula for the force on a magnetic dipole due to a varying magnetic field is

$$\mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B}) \tag{1}$$

A special case of this occurs when we have a circular current loop suspended in a radially symmetric field (such as that produced at the end of a solenoid, or along the axis of another dipole). In that case, imagine that at the location of the circular loop, the field spreads outwards like the petals of a flower, so that it has a vertical component (parallel to the axis of symmetry) and a horizontal, radial component, perpendicular to the axis. The force due to the field on the current is

$$\mathbf{F}=I\int d\ell\times\mathbf{B} \tag{2}$$

integrated around the circle. The contributions from the vertical component of $\mathbf{B}$ cancel out, but the horizontal component of magnitude $B\cos\alpha$ (where the angle between the field and the horizontal is α) adds up around the circle to give a net force (downwards) of

$$F=2\pi RIB\cos\alpha \tag{3}$$

where R is the radius of the circular loop.

We can apply both these formulas to the case of two magnetic dipoles $\mathbf{m}_1$ and $\mathbf{m}_2$ oriented along the z axis (so their axes are parallel) and separated by a distance r. First, we use the gradient formula at the top. We start with the dipole field of $\mathbf{m}_1$ in spherical coordinates:

$$\mathbf{B}=\frac{\mu_0 m_1}{4\pi r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right) \tag{4}$$

Since $\mathbf{m}_2=m_2\hat{\mathbf{z}}$, we get

$$\mathbf{m}_2\cdot\mathbf{B}=\frac{\mu_0 m_1 m_2}{4\pi r^3}\left(2\cos^2\theta-\sin^2\theta\right) \tag{5}$$

Taking the gradient in spherical coordinates, we get

$$\begin{align}
\mathbf{F} &= \nabla\left(\mathbf{m}_2\cdot\mathbf{B}\right) \tag{6}\\
&= \frac{\mu_0 m_1 m_2}{4\pi r^4}\left[-3\left(2\cos^2\theta-\sin^2\theta\right)\hat{\mathbf{r}}-6\sin\theta\cos\theta\hat{\boldsymbol{\theta}}\right] \tag{7}
\end{align}$$

In the case of an infinitesimal dipole on the z axis, $\theta=0$ and we get

$$\begin{align}
\mathbf{F} &= -\frac{3\mu_0 m_1 m_2}{2\pi r^4}\hat{\mathbf{r}} \tag{8}\\
&= -\frac{3\mu_0 m_1 m_2}{2\pi r^4}\hat{\mathbf{z}} \tag{9}
\end{align}$$

That is, the dipole $\mathbf{m}_2$ feels an attractive force towards $\mathbf{m}_1$.

Doing the same calculation with the seemingly simpler formula 3 is actually more complicated, mainly because we're dealing with an ideal, and thus infinitesimal, dipole rather than a finite circular current loop. We can start by working out the field due to $\mathbf{m}_1$ at the location of a finite loop with radius R. Here, we'll redefine r as the distance from $\mathbf{m}_1$ to the centre of the loop, so the field on the loop itself is then

$$\mathbf{B}=\frac{\mu_0 m_1}{4\pi\rho^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right) \tag{10}$$

where

$$\begin{align}
\rho &\equiv \sqrt{r^2+R^2} \tag{11}\\
\cos\theta &= \frac{r}{\rho} \tag{12}\\
\sin\theta &= \frac{R}{\rho} \tag{13}
\end{align}$$

We need the horizontal component of $\mathbf{B}$ at the loop, which we can get by considering the r and θ components separately. The horizontal component of B_r is

$$\begin{align}
B_{rh} &= \frac{\mu_0 m_1}{4\pi\rho^3}\left(2\cos\theta\cos\left(\frac{\pi}{2}-\theta\right)\right) \tag{14}\\
&= \frac{\mu_0 m_1}{4\pi\rho^3}\left(2\cos\theta\sin\theta\right) \tag{15}
\end{align}$$

Similarly, the horizontal component of B_θ is

$$B_{\theta h} = \frac{\mu_0 m_1}{4\pi\rho^3}(\sin\theta\cos\theta) \tag{16}$$

The total horizontal component of the field is the sum of these two, and replaces the $B\cos\alpha$ factor in 3:

$$B_h = \frac{3\mu_0 m_1}{4\pi\rho^3}\sin\theta\cos\theta \tag{17}$$

Now we need to express the $2\pi RI$ factor in terms of m_2. The magnetic moment of the loop is $m_2 = \pi R^2 I$, so

$$2\pi RI = \frac{2m_2}{R} \tag{18}$$

and the force on the loop is then

$$F = \frac{3\mu_0 m_1 m_2}{2\pi\rho^3 R}\sin\theta\cos\theta \tag{19}$$

To get the formula for an infinitesimal loop, we need to take the limit as $R \to 0$. In this limit $\rho \to r$, $\cos\theta \to 1$ and $\sin\theta \to R/r$ so we get

$$\lim_{R\to 0} F = \frac{3\mu_0 m_1 m_2}{2\pi r^4} \tag{20}$$

which agrees (in magnitude) with the formula obtained above by the gradient method.

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