TOROIDAL SOLENOID WITH A GAP

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.10.

As another example of how we can calculate the magnetic field due to the magnetization of an object, suppose we start with an iron bar with square cross-section of side length a and length L, and having a constant magnetization $\mathbf{M}$ parallel to the length of the bar. We now bend the bar into a circle so that the end points almost touch, but are separated by a gap of width w. Our task is to find the magnetic field at the centre of the gap.

The first thing to recognize is that by bending the bar into a circle, the magnetization is no longer constant in magnitude within the bar, since the outer edge of the circle will be stretched relative to the inside edge. The new magnetization will be

$$M'(r) = \frac{L}{2\pi r} M \tag{1}$$

We can verify this by finding the magnitude of the total dipole moment in the bar:

$$\int_{L/2\pi}^{L/2\pi+a} 2\pi r a M'(r)\, dr = a^2 L M \tag{2}$$

so the total magnetic moment is the same as in the original unbent bar. Since the magnetization now points along the circle, its vector form is

$$\mathbf{M}' = \frac{L}{2\pi r} M \hat{\phi} \tag{3}$$

From the formula for the curl in cylindrical coordinates, we then get for the bound volume current

$$\mathbf{J}_b = \nabla \times \mathbf{M}' = 0 \tag{4}$$

The surface current has the values on the four sides of the square:

$$\mathbf{K}_b = \begin{cases} M\hat{\mathbf{z}} & \text{inside edge} \\ \frac{LM}{2\pi r}\hat{\mathbf{r}} & \text{top edge} \\ -\frac{ML}{(L+2\pi a)}\hat{\mathbf{z}} & \text{outside edge} \\ -\frac{LM}{2\pi r}\hat{\mathbf{r}} & \text{bottom edge} \end{cases} \tag{5}$$

If we assume that $L \gg a$ so that $r \approx L/2\pi$ for the whole width of the rod, we can approximate $\mathbf{K}_b$ as a current density of magnitude M on all four sides, so we have essentially a torus-shaped solenoid with a square cross section. Griffiths works out the field inside a toroidal solenoid of arbitrary cross-section as Example 5.10, using methods similar to those we used for a linear solenoid, with the result (translating to the quantities used in this post):

$$\mathbf{B}_{torus} = \frac{\mu_0 ML}{2\pi(L/2\pi)}\hat{\phi} \tag{6}$$

$$= \mu_0 M\hat{\phi} \tag{7}$$

This is the field inside a complete torus. To handle the gap, we can approximate it by a square current loop with current opposing that in the torus. We've worked out the field at the centre of such a loop before, so again translating this into the quantities used here, we have:

$$\mathbf{B}_{loop} = -\frac{2\sqrt{2}Mw\mu_0}{\pi a}\hat{\phi} \tag{8}$$

The net field at the centre of the gap is the sum of these two:

$$\mathbf{B} = \mu_0 M\hat{\phi}\left(1 - \frac{2\sqrt{2}w}{\pi a}\right) \tag{9}$$