MAGNETIZATION: MICROSCOPIC VERSUS MACROSCOPIC

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.11.

An object containing a magnetization distribution gives rise to a vector potential according to:

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\mathbf{M}\left(\mathbf{r}'\right)\times\left(\mathbf{r}-\mathbf{r}'\right)d^3\mathbf{r}' \tag{0.1}$$

This formula treats the magnetization as a continuous vector field, whereas in reality, it is due to many discrete current loops created by spinning and orbiting electrons and atomic nuclei. Even if the macroscopic currents are constant, the microscopic configuration of the currents still changes rapidly, so the microscopic magnetization is a rapidly varying function. Thus the formula really relies on $\mathbf{M}$ being an average magnetization over some region around the observation point.

However, there appears to be something of a contradiction here, since this formula was derived from the formula for a single, ideal dipole

$$\mathbf{A}=\frac{\mu_0}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|^3}\mathbf{m}\times\left(\mathbf{r}-\mathbf{r}'\right) \tag{0.2}$$

In other words, we applied a microscopic formula to derive a macroscopic one. We can reconcile the two situations by the following argument.

We define the macroscopic magnetic field at a point $\mathbf{r}$ within the object as the sum of the average field $\mathbf{B}_{out}$ due to all currents outside a sphere of radius R centred at $\mathbf{r}$ and the field $\mathbf{B}_{in}$ produced by the currents within the same sphere:

$$\mathbf{B}=\mathbf{B}_{out}+\mathbf{B}_{in} \tag{0.3}$$

For the exterior currents, we can assume that these currents are far enough away that only their average effect is felt at the observation point, so we can use the formula above to find the potential of these exterior currents within the sphere:

$$\mathbf{A}_{out}(\mathbf{r}_{in}) = \frac{\mu_0}{4\pi}\int_{out}\frac{1}{|\mathbf{r}_{in}-\mathbf{r}'|^3}\mathbf{M}(\mathbf{r}')\times(\mathbf{r}_{in}-\mathbf{r}')\,d^3\mathbf{r}' \tag{0.4}$$

where the integral is taken over points $\mathbf{r}'$ outside the sphere, and $\mathbf{r}_{in}$ represents a point inside the sphere. So in principle, we need to average this formula over all points $\mathbf{r}_{in}$ inside the sphere. However, we showed earlier that the average field within the sphere due to any current distribution outside a sphere is equal to the exact field produced at the centre of the sphere. Thus the average of the field produced by these external currents is just

$$\mathbf{A}_{av\ out} = \frac{\mu_0}{4\pi}\int_{out}\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')\,d^3\mathbf{r}' \tag{0.5}$$

where $\mathbf{r}$ now points to the centre of the sphere. The magnetic field can then be derived from this by the usual formula $\mathbf{B}_{out} = \nabla\times\mathbf{A}_{av\ out}$. Thus the field due to the external currents is given correctly by using 0.1 to calculate the field.

For the currents within the sphere, we also showed that the average magnetic field within the sphere for these currents is

$$\mathbf{B}_{av} = \frac{\mu_0}{2\pi R^3}\mathbf{m} \tag{0.6}$$

where $\mathbf{m}$ is the total magnetic moment of the currents within the sphere. Now we're assuming that although R is large compared to the scale of microscopic fluctuations, it is small enough that the magnetization $\mathbf{M}$ is essentially constant over the sphere, so that

$$\mathbf{m} = \frac{4}{3}\pi R^3\mathbf{M} \tag{0.7}$$

so the average field due to interior currents is

$$\mathbf{B}_{in} = \frac{2}{3}\mu_0\mathbf{M} \tag{0.8}$$

Griffiths shows in his example 6.1 that the internal magnetic field due to a uniformly magnetized sphere also happens to be $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$, and *that* result was derived ultimately from the formula 0.1. In other words, the average field at a point $\mathbf{r}$ as derived from the magnetization formula is the same as that derived by considering the microscopic currents, so the extension of the formula for ideal dipoles to a volume magnetization actually does give the correct macroscopic result.