AUXILIARY MAGNETIC FIELD H

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.12.

Jackson, J. D. (1999) Classical Electrodynamics, 3rd Edition; Wiley - Sections 5.8, 5.9.

An object containing a magnetization distribution can be modelled by replacing the magnetization by bound volume and surface currents $\mathbf{J}_b$ and $\mathbf{K}_b$. If we add in some extra *free* current $\mathbf{J}_f$ not due to the magnetization (for example, by plugging the object into the electric mains), then the total current at a point inside the object is

$$\mathbf{J}=\mathbf{J}_b+\mathbf{J}_f \tag{1}$$

Since $\mathbf{J}_b=\nabla\times\mathbf{M}$ by definition, we can write Ampère's law as

$$\begin{aligned}\frac{1}{\mu_0}\nabla\times\mathbf{B} &= \mathbf{J} &(2)\\ &= \mathbf{J}_f+\nabla\times\mathbf{M} &(3)\\ \nabla\times\left(\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}\right) &= \mathbf{J}_f &(4)\end{aligned}$$

The quantity in parentheses is given the symbol $\mathbf{H}$:

$$\mathbf{H}\equiv\frac{1}{\mu_0}\mathbf{B}-\mathbf{M} \tag{5}$$

$\mathbf{H}$ is sometimes called the auxiliary magnetic field or sometimes just the magnetic field, with $\mathbf{B}$ being called the magnetic flux density. This allows a variant form of Ampère's law:

$$\begin{aligned}\nabla\times\mathbf{H} &= \mathbf{J}_f &(6)\\ \oint\mathbf{H}\cdot d\ell &= I_f &(7)\end{aligned}$$

where I_f is the free current enclosed by the path of integration in the second line.

One thing that is a bit worrying is that the bound surface current $\mathbf{K}_b$ seems to have vanished in this derivation. Griffiths makes no mention of this, but Jackson gets round the problem by saying that the surface integral from which $\mathbf{K}_b$ was defined, namely $\frac{\mu_0}{4\pi}\int\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\times d\mathbf{a}'$ is zero by assuming that the magnetization $\mathbf{M}$ is well-behaved and localized, and we can take the surface at infinity where $\mathbf{M}=0$. He then states later that in some idealized problems, it is convenient to assume that $\mathbf{M}$ is discontinuous at the boundary between two objects (for example, between a magnetized object and the surrounding air), and in that case, the surface current term must be added in. However, in any 'real' physical situation, discontinuities never occur so the surface term doesn't appear and the definition of $\mathbf{H}$ above is valid.

In any case, we can use this definition of $\mathbf{H}$ to calculate $\mathbf{B}$ more easily in some idealized situations. For example, if we have an infinitely long cylinder of radius R with a fixed $\mathbf{M}=kr\hat{\mathbf{z}}$, we can find $\mathbf{B}$ by two methods.

First, we use the bound current approach.

$$\mathbf{J}_b = \nabla\times\mathbf{M} \tag{8}$$
$$= -k\hat{\phi} \tag{9}$$
$$\mathbf{K}_b = \mathbf{M}(R)\times\hat{\mathbf{n}} \tag{10}$$
$$= kR\hat{\phi} \tag{11}$$

Note that the total bound current is zero, since the total volume current is

$$\int_0^R \mathbf{J}_b dr = -kR\hat{\phi} \tag{12}$$

Both bound currents effectively produce solenoids, so the field outside the cylinder is zero. Inside, we have field due to the surface current, which is

$$\mathbf{B}_K = \mu_0 kR\hat{\mathbf{z}} \tag{13}$$

and we must add to that the field due to those parts of the cylinder with a radius greater than the the radius r of interest. The total current outside radius r is

$$\int_r^R \mathbf{J}_b dr = -k(R-r)\hat{\phi} \tag{14}$$

so the total field is

$$\mathbf{B} = \mu_0 kR\hat{\mathbf{z}} - \mu_0 k(R-r)\hat{\mathbf{z}} = \mu_0 kr\hat{\mathbf{z}} = \mu_0\mathbf{M} \tag{15}$$

Using $\mathbf{H}$, we can take a loop of integration of radius r centred on the z axis. Since there is no free current, we get

$$\oint \mathbf{H} \cdot d\ell = 0 \tag{16}$$

and from the symmetry of the problem we can conclude that $\mathbf{H} = 0$.

Alternatively, we can work from the curl equation which gives $\nabla \times \mathbf{H} = 0$. This on its own isn't enough to conclude that $\mathbf{H} = 0$, but we can also calculate the divergence

$$\nabla \cdot \mathbf{H} = \frac{1}{\mu_0} \nabla \cdot \mathbf{B} - \nabla \cdot \mathbf{M} \tag{17}$$

Since $\nabla \cdot \mathbf{B} = 0$ always, and by direct calculation we can show that $\nabla \cdot \mathbf{M} = 0$ in this case, we have both the curl and divergence of $\mathbf{H}$ as zero, so $\mathbf{H}$ must be zero. From that, we can conclude immediately from the definition of $\mathbf{H}$ that $\mathbf{B} = \mu_0 \mathbf{M}$.

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