

MAGNETOSTATIC BOUNDARY CONDITIONS; LAPLACE'S EQUATION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.15.

From the magnetostatic boundary conditions on the magnetic field \mathbf{B} we can work out the boundary conditions on the auxiliary field \mathbf{H} . First, we need the divergence of \mathbf{H} :

$$(0.1) \quad \nabla \cdot \mathbf{H} = \frac{1}{\mu_0} \nabla \cdot \mathbf{B} - \nabla \cdot \mathbf{M} = -\nabla \cdot \mathbf{M}$$

Using a similar argument to that for the electric field, we can see that \mathbf{H} and \mathbf{M} have equal and opposite discontinuities at a boundary, so

$$(0.2) \quad H_{\perp}^{above} - H_{\perp}^{below} = -\left(M_{\perp}^{above} - M_{\perp}^{below}\right)$$

Using the same argument as for magnetostatic boundary conditions, the boundary condition on the component of \mathbf{H} parallel to the boundary is

$$(0.3) \quad \mathbf{H}_{\parallel}^{above} - \mathbf{H}_{\parallel}^{below} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

where \mathbf{K}_f is the free surface current density on the boundary.

The quantity \mathbf{H}_{\parallel} is written as a vector since it lies in the tangent plane to the surface and has a direction given by the cross product $\mathbf{K}_f \times \hat{\mathbf{n}}$.

In the special case where there is no free current anywhere, then

$$(0.4) \quad \nabla \times \mathbf{H} = \mathbf{J}_f = 0$$

which means that the vector field \mathbf{H} can be written as the gradient of a scalar field W :

$$(0.5) \quad \mathbf{H} = -\nabla W$$

As an example of using this, we can derive the field due to a uniformly magnetized sphere. The magnetization is $\mathbf{M} = M\hat{\mathbf{z}}$ within the sphere and zero outside. Since \mathbf{M} is constant everywhere except at the boundary, then

$$(0.6) \quad \nabla \cdot \mathbf{H} = -\nabla^2 W = 0$$

everywhere except at the boundary. This is Laplace's equation and in spherical coordinates, the general solution is

$$(0.7) \quad W(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

We can follow the usual procedure for finding the coefficients by imposing boundary conditions on W . Just as in the electrostatic case, the potential must be continuous at the boundary, and must be finite everywhere. This means that inside the sphere, we must have $B_l = 0$ and outside the sphere $A_l = 0$ so that

$$(0.8) \quad W_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$(0.9) \quad W_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

At the boundary we must have $W_{in} = W_{out}$ and equating coefficients of each Legendre polynomial we get

$$(0.10) \quad B_l = A_l R^{2l+1}$$

We can now consider the derivative of W in the r direction, which gives the normal component of \mathbf{H} as $-\partial W / \partial r$. Using 0.2 we have

$$(0.11) \quad \left. \frac{\partial W}{\partial r} \right|_{out} - \left. \frac{\partial W}{\partial r} \right|_{in} = M_{\perp}^{above} - M_{\perp}^{below}$$

$$(0.12) \quad = -M\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}$$

$$(0.13) \quad = -M \cos \theta$$

Taking the derivatives of the series above, we get for $l = 1$ after using 0.10

$$(0.14) \quad -\frac{l+1}{R^{l+2}}B_l = lA_lR^{l-1} - M$$

$$(0.15) \quad A_1 = \frac{M}{3}$$

$$(0.16) \quad B_1 = \frac{M}{3}R^3$$

For $l \neq 1$ we have

$$(0.17) \quad -\frac{l+1}{R^{l+2}}B_l = lA_lR^{l-1}$$

$$(0.18) \quad -(l+1)R^{l-1}A_l = lA_lR^{l-1}$$

$$(0.19) \quad (2l+1)A_l = 0$$

Since this must be true for all $l \neq 1$ we must have $A_l = B_l = 0$ for these cases. Thus

$$(0.20) \quad W_{in} = \frac{M}{3}r \cos \theta$$

$$(0.21) \quad W_{out} = \frac{MR^3}{3r^2} \cos \theta$$

Taking the negative gradient, we get, since $r \cos \theta = z$

$$(0.22) \quad \mathbf{H}_{in} = -\frac{M}{3}\hat{\mathbf{z}}$$

$$(0.23) \quad \mathbf{H}_{out} = \frac{MR^3}{3r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

From this we can get the field $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$ (remember $\mathbf{M} = 0$ outside the sphere):

$$(0.24) \quad \mathbf{B}_{in} = \frac{2\mu_0 M}{3}\hat{\mathbf{z}}$$

$$(0.25) \quad \mathbf{B}_{out} = \frac{\mu_0 MR^3}{3r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

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