

MAGNETOSTATIC BOUNDARY CONDITIONS; LAPLACE'S EQUATION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.15.

From the magnetostatic boundary conditions on the magnetic field \mathbf{B} we can work out the boundary conditions on the auxiliary field \mathbf{H} . First, we need the divergence of \mathbf{H} :

$$\nabla \cdot \mathbf{H} = \frac{1}{\mu_0} \nabla \cdot \mathbf{B} - \nabla \cdot \mathbf{M} = -\nabla \cdot \mathbf{M} \quad (1)$$

Using a similar argument to that for the electric field, we can see that \mathbf{H} and \mathbf{M} have equal and opposite discontinuities at a boundary, so

$$H_{\perp}^{above} - H_{\perp}^{below} = -\left(M_{\perp}^{above} - M_{\perp}^{below}\right) \quad (2)$$

Using the same argument as for magnetostatic boundary conditions, the boundary condition on the component of \mathbf{H} parallel to the boundary is

$$\mathbf{H}_{\parallel}^{above} - \mathbf{H}_{\parallel}^{below} = \mathbf{K}_f \times \hat{\mathbf{n}} \quad (3)$$

where \mathbf{K}_f is the free surface current density on the boundary.

The quantity \mathbf{H}_{\parallel} is written as a vector since it lies in the tangent plane to the surface and has a direction given by the cross product $\mathbf{K}_f \times \hat{\mathbf{n}}$.

In the special case where there is no free current anywhere, then

$$\nabla \times \mathbf{H} = \mathbf{J}_f = 0 \quad (4)$$

which means that the vector field \mathbf{H} can be written as the gradient of a scalar field W :

$$\mathbf{H} = -\nabla W \quad (5)$$

As an example of using this, we can derive the field due to a uniformly magnetized sphere. The magnetization is $\mathbf{M} = M\hat{\mathbf{z}}$ within the sphere and zero outside. Since \mathbf{M} is constant everywhere except at the boundary, then

$$\nabla \cdot \mathbf{H} = -\nabla^2 W = 0 \quad (6)$$

everywhere except at the boundary. This is Laplace's equation and in spherical coordinates, the general solution is

$$W(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) \quad (7)$$

We can follow the usual procedure for finding the coefficients by imposing boundary conditions on W . Just as in the electrostatic case, the potential must be continuous at the boundary, and must be finite everywhere. This means that inside the sphere, we must have $B_l = 0$ and outside the sphere $A_l = 0$ so that

$$W_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \quad (8)$$

$$W_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) \quad (9)$$

At the boundary we must have $W_{in} = W_{out}$ and equating coefficients of each Legendre polynomial we get

$$B_l = A_l R^{2l+1} \quad (10)$$

We can now consider the derivative of W in the r direction, which gives the normal component of \mathbf{H} as $-\partial W / \partial r$. Using 2 we have

$$\left. \frac{\partial W}{\partial r} \right|_{out} - \left. \frac{\partial W}{\partial r} \right|_{in} = M_{\perp}^{above} - M_{\perp}^{below} \quad (11)$$

$$= -M \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} \quad (12)$$

$$= -M \cos \theta \quad (13)$$

Taking the derivatives of the series above, we get for $l = 1$ after using 10

$$-\frac{l+1}{R^{l+2}} B_l = l A_l R^{l-1} - M \quad (14)$$

$$A_1 = \frac{M}{3} \quad (15)$$

$$B_1 = \frac{M}{3} R^3 \quad (16)$$

For $l \neq 1$ we have

$$-\frac{l+1}{R^{l+2}}B_l = lA_lR^{l-1} \quad (17)$$

$$-(l+1)R^{l-1}A_l = lA_lR^{l-1} \quad (18)$$

$$(2l+1)A_l = 0 \quad (19)$$

Since this must be true for all $l \neq 1$ we must have $A_l = B_l = 0$ for these cases. Thus

$$W_{in} = \frac{M}{3}r \cos \theta \quad (20)$$

$$W_{out} = \frac{MR^3}{3r^2} \cos \theta \quad (21)$$

Taking the negative gradient, we get, since $r \cos \theta = z$

$$\mathbf{H}_{in} = -\frac{M}{3}\hat{\mathbf{z}} \quad (22)$$

$$\mathbf{H}_{out} = \frac{MR^3}{3r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) \quad (23)$$

From this we can get the field $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ (remember $\mathbf{M} = 0$ outside the sphere):

$$\mathbf{B}_{in} = \frac{2\mu_0 M}{3}\hat{\mathbf{z}} \quad (24)$$

$$\mathbf{B}_{out} = \frac{\mu_0 MR^3}{3r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) \quad (25)$$

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