MAGNETOSTATIC BOUNDARY CONDITIONS; LAPLACE'S EQUATION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.15.

From the magnetostatic boundary conditions on the magnetic field $\mathbf{B}$ we can work out the boundary conditions on the auxiliary field $\mathbf{H}$. First, we need the divergence of $\mathbf{H}$:

$$\nabla\cdot\mathbf{H}=\frac{1}{\mu_0}\nabla\cdot\mathbf{B}-\nabla\cdot\mathbf{M}=-\nabla\cdot\mathbf{M} \tag{1}$$

Using a similar argument to that for the electric field, we can see that $\mathbf{H}$ and $\mathbf{M}$ have equal and opposite discontinuites at a boundary, so

$$H_{\perp}^{above}-H_{\perp}^{below}=-\left(M_{\perp}^{above}-M_{\perp}^{below}\right) \tag{2}$$

Using the same argument as for magnetostatic boundary conditions, the boundary condition on the component of $\mathbf{H}$ parallel to the boundary is

$$\mathbf{H}_{\parallel}^{above}-\mathbf{H}_{\parallel}^{below}=\mathbf{K}_f\times\hat{\mathbf{n}} \tag{3}$$

where $\mathbf{K}_f$ is the free surface current density on the boundary.

The quantity $\mathbf{H}_{\parallel}$ is written as a vector since it lies in the tangent plane to the surface and has a direction given by the cross product $\mathbf{K}_f\times\hat{\mathbf{n}}$.

In the special case where there is no free current anywhere, then

$$\nabla\times\mathbf{H}=\mathbf{J}_f=0 \tag{4}$$

which means that the vector field $\mathbf{H}$ can be written as the gradient of a scalar field W:

$$\mathbf{H}=-\nabla W \tag{5}$$

As an example of using this, we can derive the field due to a uniformly magnetized sphere. The magnetization is $\mathbf{M} = M\hat{\mathbf{z}}$ within the sphere and zero outside. Since $\mathbf{M}$ is constant everywhere except at the boundary, then

$$\nabla \cdot \mathbf{H} = -\nabla^2 W = 0 \tag{6}$$

everywhere except at the boundary. This is Laplace's equation and in spherical coordinates, the general solution is

$$W(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \tag{7}$$

We can follow the usual procedure for finding the coefficients by imposing boundary conditions on W. Just as in the electrostatic case, the potential must be continuous at the boundary, and must be finite everywhere. This means that inside the sphere, we must have $B_l = 0$ and outside the sphere $A_l = 0$ so that

$$W_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{8}$$

$$W_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{9}$$

At the boundary we must have $W_{in} = W_{out}$ and equating coefficients of each Legendre polynomial we get

$$B_l = A_l R^{2l+1} \tag{10}$$

We can now consider the derivative of W in the r direction, which gives the normal component of $\mathbf{H}$ as $-\partial W/\partial r$. Using 2 we have

$$\left.\frac{\partial W}{\partial r}\right|_{out} - \left.\frac{\partial W}{\partial r}\right|_{in} = M_{\perp}^{above} - M_{\perp}^{below} \tag{11}$$

$$= -M\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} \tag{12}$$

$$= -M\cos\theta \tag{13}$$

Taking the derivatives of the series above, we get for $l = 1$ after using 10

$$-\frac{l+1}{R^{l+2}}B_l = lA_lR^{l-1}-M \tag{14}$$

$$A_1 = \frac{M}{3} \tag{15}$$

$$B_1 = \frac{M}{3}R^3 \tag{16}$$

For $l\neq1$ we have

$$-\frac{l+1}{R^{l+2}}B_l = lA_lR^{l-1} \tag{17}$$

$$-(l+1)R^{l-1}A_l = lA_lR^{l-1} \tag{18}$$

$$(2l+1)A_l = 0 \tag{19}$$

Since this must be true for all $l\neq1$ we must have $A_l=B_l=0$ for these cases. Thus

$$W_{in} = \frac{M}{3}r\cos\theta \tag{20}$$

$$W_{out} = \frac{M}{3}\frac{R^3}{r^2}\cos\theta \tag{21}$$

Taking the negative gradient, we get, since $r\cos\theta=z$

$$\mathbf{H}_{in} = -\frac{M}{3}\hat{\mathbf{z}} \tag{22}$$

$$\mathbf{H}_{out} = \frac{MR^3}{3r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right) \tag{23}$$

From this we can get the field $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$ (remember $\mathbf{M}=0$ outside the sphere):

$$\mathbf{B}_{in} = \frac{2\mu_0M}{3}\hat{\mathbf{z}} \tag{24}$$

$$\mathbf{B}_{out} = \frac{\mu_0MR^3}{3r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right) \tag{25}$$

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