ENERGY OF MAGNETIC DIPOLE IN MAGNETIC FIELD

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.21.

Using similar techniques to those for finding the energy of an electric dipole in an electric field, we can work out the energy of a magnetic dipole in a magnetic field. The torque exerted by a field on a dipole is

$$\mathbf{N}=\mathbf{m}\times\mathbf{B} \tag{1}$$

If we bring in a dipole from infinity in such a way that no work is done (by moving the dipole along a path that is always perpendicular to the force exerted by the field) then when we arrive at the desired location, we must rotate the dipole into its final position, and this requires work to be done against the torque. If we start with $\mathbf{m}\perp\mathbf{B}$, then the work done is

$$U=mB\int_{\pi/2}^{\theta}\sin\theta d\theta=-mB\cos\theta=-\mathbf{m}\cdot\mathbf{B} \tag{2}$$

The choice of $\theta=\pi/2$ as the zero point for the energy appears to be arbitrary, although it does make the situation more symmetric. The maximum energy of mB occurs when $\mathbf{m}$ is anti-parallel to $\mathbf{B}$ and the minimum energy of $-mB$ when $\mathbf{m}$ and $\mathbf{B}$ are parallel. If we *had* chosen a different zero angle, the *difference* in energy between two angles would still be the same, and that is all that matters physically.

The interaction energy of two dipoles can be written down from the formula for the field due to a dipole:

$$\mathbf{B}_1=\frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}_1\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{m}_1\right] \tag{3}$$

We then get

$$\begin{aligned}U_{12} &= -\mathbf{m}_2\cdot\mathbf{B}_1 &(4)\\ &= \frac{\mu_0}{4\pi r^3}\left[\mathbf{m}_1\cdot\mathbf{m}_2-3\left(\mathbf{m}_1\cdot\hat{\mathbf{r}}\right)\left(\mathbf{m}_2\cdot\hat{\mathbf{r}}\right)\right] &(5)\end{aligned}$$

Here $\hat{\mathbf{r}}$ is the unit vector pointing from $\mathbf{m}_1$ to $\mathbf{m}_2$, so if we define θ_i to be the angle between $\hat{\mathbf{r}}$ and $\mathbf{m}_i$ then

$$U_{12}=\frac{\mu_0 m_1 m_2}{4\pi r^3}\left(\cos\left(\theta_1-\theta_2\right)-3\cos\theta_1\cos\theta_2\right) \tag{6}$$

The stable configuration is when this energy is a minimum, so considering the trig functions alone, we have

$$\begin{aligned}\frac{\partial U_{12}}{\partial\theta_1} &= -\sin\left(\theta_1-\theta_2\right)+3\sin\theta_1\cos\theta_2=0 && (7)\\ \frac{\partial U_{12}}{\partial\theta_2} &= \sin\left(\theta_1-\theta_2\right)+3\cos\theta_1\sin\theta_2=0 && (8)\end{aligned}$$

Using the shorthand notation $s_{12}\equiv\sin\left(\theta_1-\theta_2\right)$, $s_1\equiv\sin\theta_1$, $c_1\equiv\cos\theta_1$, etc, we have

$$\begin{aligned}s_{12} &= 3s_1c_2 && (9)\\ s_1c_2+c_1s_2 &= 0 && (10)\\ \tan\theta_1 &= -\tan\theta_2 && (11)\end{aligned}$$

Since $0\le\theta_1,\theta_2\le\pi$, this means that either $\tan\theta_1=\tan\theta_2=0$ or $\theta_1=\frac{\pi}{2}+x;\ \theta_2=\frac{\pi}{2}-x$. Either way we have $s_1=s_2\equiv s$ and $c_1=-c_2\equiv c$, so

$$\begin{aligned}s_{12} &= -3sc && (12)\\ s_1c_2-c_1s_2 &= -3sc && (13)\\ -sc-cs &= -3sc && (14)\\ -2sc &= -3sc && (15)\end{aligned}$$

The only way the last equation can be true is if $s=0$ and/or $c=0$, which means the choices are $\theta_{1,2}=0,\frac{\pi}{2},\pi$. We can try the various possibilities, but it's fairly obvious anyway that the stable configuration is when the two moments are parallel, that is, $\theta_1=\theta_2=0$ (or $\theta_1=\theta_2=\pi$). If we line up a sequence of compass needles, they will align themselves so that the north end of one points to the south end of the next, and so forth.

PINGBACKS

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