FORCE ON A MAGNETIC DIPOLE - A BETTER DERIVATION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.22.

We've seen a rather crude derivation of the force on a magnetic dipole in a varying magnetic field, which gives the result

$$\mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B}) \tag{1}$$

In that derivation, we considered a particular geometry of dipole (a square) and constrained the orientation of the dipole within the coordinate system. A more general approach is given here.

If $\mathbf{B}(\mathbf{r})$ is a general vector field, we can write its value near the location $\mathbf{r}_0$ of the dipole as a 3-d Taylor expansion to first order:

$$\mathbf{B}(\mathbf{r})=\mathbf{B}(\mathbf{r}_0)+[(\mathbf{r}-\mathbf{r}_0)\cdot\nabla]\mathbf{B}_0 \tag{2}$$

where the subscript 0 on $\mathbf{B}_0$ in the second term indicates that the derivatives of $\mathbf{B}$ are evaluated at $\mathbf{r}_0$.

Putting this into the Lorentz force law, we have (given the current I producing the dipole)

$$\begin{aligned}
\mathbf{F} &= I\oint d\ell\times\mathbf{B}(\mathbf{r}) && (3)\\
&= I\oint d\ell\times\mathbf{B}(\mathbf{r}_0)+I\oint d\ell\times[(\mathbf{r}-\mathbf{r}_0)\cdot\nabla]\mathbf{B}_0 && (4)
\end{aligned}$$

The first integral and $\mathbf{r}_0$ term in the second integral come out to zero, since we are integrating a constant around a closed loop. Thus we are left with

$$\mathbf{F}=I\oint d\ell\times(\mathbf{r}\cdot\nabla)\mathbf{B}_0 \tag{5}$$

We can write the cross product in rectangular coordinates using the Levi-Civita symbol $\varepsilon_{ijk}=+1$ for a cyclic permutation of 1,2,3, -1 for an anticyclic permutation and zero if any two indices are equal. In general

$$(\mathbf{A}\times\mathbf{B})_i=\sum_{j,k=1}^{3}\varepsilon_{ijk}A_jB_k \tag{6}$$

The dot product can be written as

$$\mathbf{A}\cdot\mathbf{B}=\sum_{l=1}^{3}A_lB_l \tag{7}$$

Using these two sums, we have for a component of the force:

$$F_i=I\sum_{j,k,l}\varepsilon_{ijk}\oint r_l d\ell_j\nabla_l B_{k0} \tag{8}$$

The integral is of a form that we've encountered when discussing the vector area **a** of a curve. For a constant vector **c**:

$$\oint \mathbf{c}\cdot\mathbf{r}d\ell=\mathbf{a}\times\mathbf{c} \tag{9}$$

In components, this is

$$\sum_l\oint r_lc_ld\ell_j=\sum_{m,n}\varepsilon_{jmn}a_mc_n \tag{10}$$

Using $\mathbf{c}=\nabla B_{k0}$, we can plug this back into the force equation to get

$$F_i=I\sum_{j,k,m,n}\varepsilon_{ijk}\varepsilon_{jmn}a_m\left(\nabla B_{k0}\right)_n \tag{11}$$

We can now use an identity for a sum over the Levi-Civita symbols:

$$\sum_j\varepsilon_{ijk}\varepsilon_{njm}=\delta_{in}\delta_{km}-\delta_{im}\delta_{kn} \tag{12}$$

Because of the cyclic property of the ε_{njm}, we have $\varepsilon_{njm}=\varepsilon_{jmn}$, so we now get for the force:

$$F_i = I\sum_{k,m,n}\left(\delta_{in}\delta_{km}-\delta_{im}\delta_{kn}\right)a_m\left(\nabla B_{k0}\right)_n \tag{13}$$

$$= I\sum_k a_k\left(\nabla B_{k0}\right)_i-I\sum_k a_i\left(\nabla B_{k0}\right)_k \tag{14}$$

For a fixed dipole, the moment is a constant and is

$$\mathbf{m} = I\mathbf{a} \tag{15}$$

so the first term is

$$I\sum_k a_k (\nabla B_{k0})_i = \sum_k m_k (\nabla B_{k0})_i \tag{16}$$

$$= \sum_k \nabla (m_k B_{k0})_i \tag{17}$$

$$= \nabla (\mathbf{m} \cdot \mathbf{B}_0)_i \tag{18}$$

In the second term, the sum comes out to

$$I\sum_k a_i (\nabla B_{k0})_k = Ia_i \sum_k (\nabla B_{k0})_k \tag{19}$$

$$= Ia_i \sum_k \frac{\partial B_{k0}}{\partial x_k} \tag{20}$$

$$= Ia_i \nabla \cdot \mathbf{B}_0 \tag{21}$$

$$= 0 \tag{22}$$

since $\nabla \cdot \mathbf{B} = 0$ in general.

Thus we get the previous result

$$\mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B}) \tag{23}$$