

FORCE ON A MAGNETIC DIPOLE - A BETTER DERIVATION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.22.

We've seen a rather crude derivation of the force on a magnetic dipole in a varying magnetic field, which gives the result

$$(0.1) \quad \mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B})$$

In that derivation, we considered a particular geometry of dipole (a square) and constrained the orientation of the dipole within the coordinate system. A more general approach is given here.

If $\mathbf{B}(\mathbf{r})$ is a general vector field, we can write its value near the location \mathbf{r}_0 of the dipole as a 3-d Taylor expansion to first order:

$$(0.2) \quad \mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{r}_0) + [(\mathbf{r} - \mathbf{r}_0) \cdot \nabla] \mathbf{B}_0$$

where the subscript 0 on \mathbf{B}_0 in the second term indicates that the derivatives of \mathbf{B} are evaluated at \mathbf{r}_0 .

Putting this into the Lorentz force law, we have (given the current I producing the dipole)

$$(0.3) \quad \mathbf{F} = I \oint d\mathbf{l} \times \mathbf{B}(\mathbf{r})$$

$$(0.4) \quad = I \oint d\mathbf{l} \times \mathbf{B}(\mathbf{r}_0) + I \oint d\mathbf{l} \times [(\mathbf{r} - \mathbf{r}_0) \cdot \nabla] \mathbf{B}_0$$

The first integral and \mathbf{r}_0 term in the second integral come out to zero, since we are integrating a constant around a closed loop. Thus we are left with

$$(0.5) \quad \mathbf{F} = I \oint d\mathbf{l} \times (\mathbf{r} \cdot \nabla) \mathbf{B}_0$$

We can write the cross product in rectangular coordinates using the Levi-Civita symbol $\epsilon_{ijk} = +1$ for a cyclic permutation of 1,2,3, -1 for an anti-cyclic permutation and zero if any two indices are equal. In general

$$(0.6) \quad (\mathbf{A} \times \mathbf{B})_i = \sum_{j,k=1}^3 \varepsilon_{ijk} A_j B_k$$

The dot product can be written as

$$(0.7) \quad \mathbf{A} \cdot \mathbf{B} = \sum_{l=1}^3 A_l B_l$$

Using these two sums, we have for a component of the force:

$$(0.8) \quad F_i = I \sum_{j,k,l} \varepsilon_{ijk} \oint r_l d\ell_j \nabla_l B_{k0}$$

The integral is of a form that we've encountered when discussing the vector area \mathbf{a} of a curve. For a constant vector \mathbf{c} :

$$(0.9) \quad \oint \mathbf{c} \cdot \mathbf{r} d\ell = \mathbf{a} \times \mathbf{c}$$

In components, this is

$$(0.10) \quad \sum_l \oint r_l c_l d\ell_j = \sum_{m,n} \varepsilon_{jmn} a_m c_n$$

Using $\mathbf{c} = \nabla B_{k0}$, we can plug this back into the force equation to get

$$(0.11) \quad F_i = I \sum_{j,k,m,n} \varepsilon_{ijk} \varepsilon_{jmn} a_m (\nabla B_{k0})_n$$

We can now use an identity for a sum over the Levi-Civita symbols:

$$(0.12) \quad \sum_j \varepsilon_{ijk} \varepsilon_{njm} = \delta_{in} \delta_{km} - \delta_{im} \delta_{kn}$$

Because of the cyclic property of the ε_{njm} , we have $\varepsilon_{njm} = \varepsilon_{jmn}$, so we now get for the force:

$$(0.13) \quad F_i = I \sum_{k,m,n} (\delta_{in} \delta_{km} - \delta_{im} \delta_{kn}) a_m (\nabla B_{k0})_n$$

$$(0.14) \quad = I \sum_k a_k (\nabla B_{k0})_i - I \sum_k a_i (\nabla B_{k0})_k$$

For a fixed dipole, the moment is a constant and is

$$(0.15) \quad \mathbf{m} = I\mathbf{a}$$

so the first term is

$$(0.16) \quad I \sum_k a_k (\nabla B_{k0})_i = \sum_k m_k (\nabla B_{k0})_i$$

$$(0.17) \quad = \sum_k \nabla (m_k B_{k0})_i$$

$$(0.18) \quad = \nabla (\mathbf{m} \cdot \mathbf{B}_0)_i$$

In the second term, the sum comes out to

$$(0.19) \quad I \sum_k a_i (\nabla B_{k0})_k = I a_i \sum_k (\nabla B_{k0})_k$$

$$(0.20) \quad = I a_i \sum_k \frac{\partial B_{k0}}{\partial x_k}$$

$$(0.21) \quad = I a_i \nabla \cdot \mathbf{B}_0$$

$$(0.22) \quad = 0$$

since $\nabla \cdot \mathbf{B} = 0$ in general.

Thus we get the previous result

$$(0.23) \quad \mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B})$$