

FORCE ON A MAGNETIC DIPOLE - A BETTER DERIVATION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.22.

We've seen a rather crude derivation of the force on a magnetic dipole in a varying magnetic field, which gives the result

$$\mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B}) \quad (1)$$

In that derivation, we considered a particular geometry of dipole (a square) and constrained the orientation of the dipole within the coordinate system. A more general approach is given here.

If $\mathbf{B}(\mathbf{r})$ is a general vector field, we can write its value near the location \mathbf{r}_0 of the dipole as a 3-d Taylor expansion to first order:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{r}_0) + [(\mathbf{r} - \mathbf{r}_0) \cdot \nabla] \mathbf{B}_0 \quad (2)$$

where the subscript 0 on \mathbf{B}_0 in the second term indicates that the derivatives of \mathbf{B} are evaluated at \mathbf{r}_0 .

Putting this into the Lorentz force law, we have (given the current I producing the dipole)

$$\mathbf{F} = I \oint d\mathbf{l} \times \mathbf{B}(\mathbf{r}) \quad (3)$$

$$= I \oint d\mathbf{l} \times \mathbf{B}(\mathbf{r}_0) + I \oint d\mathbf{l} \times [(\mathbf{r} - \mathbf{r}_0) \cdot \nabla] \mathbf{B}_0 \quad (4)$$

The first integral and \mathbf{r}_0 term in the second integral come out to zero, since we are integrating a constant around a closed loop. Thus we are left with

$$\mathbf{F} = I \oint d\mathbf{l} \times (\mathbf{r} \cdot \nabla) \mathbf{B}_0 \quad (5)$$

We can write the cross product in rectangular coordinates using the Levi-Civita symbol $\epsilon_{ijk} = +1$ for a cyclic permutation of 1,2,3, -1 for an anti-cyclic permutation and zero if any two indices are equal. In general

$$(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k=1}^3 \epsilon_{ijk} A_j B_k \quad (6)$$

The dot product can be written as

$$\mathbf{A} \cdot \mathbf{B} = \sum_{l=1}^3 A_l B_l \quad (7)$$

Using these two sums, we have for a component of the force:

$$F_i = I \sum_{j,k,l} \epsilon_{ijk} \oint r_l dl_j \nabla_l B_{k0} \quad (8)$$

The integral is of a form that we've encountered when discussing the vector area \mathbf{a} of a curve. For a constant vector \mathbf{c} :

$$\oint \mathbf{c} \cdot \mathbf{r} dl = \mathbf{a} \times \mathbf{c} \quad (9)$$

In components, this is

$$\sum_l \oint r_l c_l dl_j = \sum_{m,n} \epsilon_{jmn} a_m c_n \quad (10)$$

Using $\mathbf{c} = \nabla B_{k0}$, we can plug this back into the force equation to get

$$F_i = I \sum_{j,k,m,n} \epsilon_{ijk} \epsilon_{jmn} a_m (\nabla B_{k0})_n \quad (11)$$

We can now use an identity for a sum over the Levi-Civita symbols:

$$\sum_j \epsilon_{ijk} \epsilon_{njm} = \delta_{in} \delta_{km} - \delta_{im} \delta_{kn} \quad (12)$$

Because of the cyclic property of the ϵ_{njm} , we have $\epsilon_{njm} = \epsilon_{jmn}$, so we now get for the force:

$$F_i = I \sum_{k,m,n} (\delta_{in} \delta_{km} - \delta_{im} \delta_{kn}) a_m (\nabla B_{k0})_n \quad (13)$$

$$= I \sum_k a_k (\nabla B_{k0})_i - I \sum_k a_i (\nabla B_{k0})_k \quad (14)$$

For a fixed dipole, the moment is a constant and is

$$\mathbf{m} = I \mathbf{a} \quad (15)$$

so the first term is

$$I \sum_k a_k (\nabla B_{k0})_i = \sum_k m_k (\nabla B_{k0})_i \quad (16)$$

$$= \sum_k \nabla (m_k B_{k0})_i \quad (17)$$

$$= \nabla (\mathbf{m} \cdot \mathbf{B}_0)_i \quad (18)$$

In the second term, the sum comes out to

$$I \sum_k a_i (\nabla B_{k0})_k = I a_i \sum_k (\nabla B_{k0})_k \quad (19)$$

$$= I a_i \sum_k \frac{\partial B_{k0}}{\partial x_k} \quad (20)$$

$$= I a_i \nabla \cdot \mathbf{B}_0 \quad (21)$$

$$= 0 \quad (22)$$

since $\nabla \cdot \mathbf{B} = 0$ in general.

Thus we get the previous result

$$\mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B}) \quad (23)$$