FORCE ON A MAGNETIC DIPOLE - A BETTER DERIVATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.22.

We've seen a rather crude derivation of the force on a magnetic dipole in a varying magnetic field, which gives the result

$$\mathbf{F}=\nabla\left(\mathbf{m}\cdot\mathbf{B}\right) \tag{1}$$

In that derivation, we considered a particular geometry of dipole (a square) and constrained the orientation of the dipole within the coordinate system. A more general approach is given here.

If $\mathbf{B}\left(\mathbf{r}\right)$ is a general vector field, we can write its value near the location $\mathbf{r}_0$ of the dipole as a 3-d Taylor expansion to first order:

$$\mathbf{B}\left(\mathbf{r}\right)=\mathbf{B}\left(\mathbf{r}_0\right)+\left[\left(\mathbf{r}-\mathbf{r}_0\right)\cdot\nabla\right]\mathbf{B}_0 \tag{2}$$

where the subscript 0 on $\mathbf{B}_0$ in the second term indicates that the derivatives of $\mathbf{B}$ are evaluated at $\mathbf{r}_0$.

Putting this into the Lorentz force law, we have (given the current I producing the dipole)

$$\begin{aligned}\mathbf{F} &= I\oint d\ell\times\mathbf{B}\left(\mathbf{r}\right) && (3)\\ &= I\oint d\ell\times\mathbf{B}\left(\mathbf{r}_0\right)+I\oint d\ell\times\left[\left(\mathbf{r}-\mathbf{r}_0\right)\cdot\nabla\right]\mathbf{B}_0 && (4)\end{aligned}$$

The first integral and $\mathbf{r}_0$ term in the second integral come out to zero, since we are integrating a constant around a closed loop. Thus we are left with

$$\mathbf{F}=I\oint d\ell\times\left(\mathbf{r}\cdot\nabla\right)\mathbf{B}_0 \tag{5}$$

We can write the cross product in rectangular coordinates using the Levi-Civita symbol $\varepsilon_{ijk}=+1$ for a cyclic permutation of 1,2,3, -1 for an anti-cyclic permutation and zero if any two indices are equal. In general

$$(\mathbf{A}\times\mathbf{B})_i=\sum_{j,k=1}^{3}\varepsilon_{ijk}A_jB_k \tag{6}$$

The dot product can be written as

$$\mathbf{A}\cdot\mathbf{B}=\sum_{l=1}^{3}A_lB_l \tag{7}$$

Using these two sums, we have for a component of the force:

$$F_i=I\sum_{j,k,l}\varepsilon_{ijk}\oint r_l d\ell_j\nabla_l B_{k0} \tag{8}$$

The integral is of a form that we've encountered when discussing the vector area $\mathbf{a}$ of a curve. For a constant vector $\mathbf{c}$:

$$\oint \mathbf{c}\cdot\mathbf{r}d\ell=\mathbf{a}\times\mathbf{c} \tag{9}$$

In components, this is

$$\sum_l\oint r_lc_ld\ell_j=\sum_{m,n}\varepsilon_{jmn}a_mc_n \tag{10}$$

Using $\mathbf{c}=\nabla B_{k0}$, we can plug this back into the force equation to get

$$F_i=I\sum_{j,k,m,n}\varepsilon_{ijk}\varepsilon_{jmn}a_m\left(\nabla B_{k0}\right)_n \tag{11}$$

We can now use an identity for a sum over the Levi-Civita symbols:

$$\sum_j\varepsilon_{ijk}\varepsilon_{njm}=\delta_{in}\delta_{km}-\delta_{im}\delta_{kn} \tag{12}$$

Because of the cyclic property of the ε_{njm}, we have $\varepsilon_{njm}=\varepsilon_{jmn}$, so we now get for the force:

$$\begin{aligned}F_i &= I\sum_{k,m,n}\left(\delta_{in}\delta_{km}-\delta_{im}\delta_{kn}\right)a_m\left(\nabla B_{k0}\right)_n &(13)\\ &= I\sum_k a_k\left(\nabla B_{k0}\right)_i-I\sum_k a_i\left(\nabla B_{k0}\right)_k &(14)\end{aligned}$$

For a fixed dipole, the moment is a constant and is

$$\mathbf{m}=I\mathbf{a} \tag{15}$$

so the first term is

$$
\begin{aligned}
I\sum_k a_k \left(\nabla B_{k0}\right)_i &= \sum_k m_k \left(\nabla B_{k0}\right)_i && (16)\\
&= \sum_k \nabla \left(m_k B_{k0}\right)_i && (17)\\
&= \nabla \left(\mathbf{m}\cdot\mathbf{B}_0\right)_i && (18)
\end{aligned}
$$

In the second term, the sum comes out to

$$
\begin{aligned}
I\sum_k a_i \left(\nabla B_{k0}\right)_k &= Ia_i \sum_k \left(\nabla B_{k0}\right)_k && (19)\\
&= Ia_i \sum_k \frac{\partial B_{k0}}{\partial x_k} && (20)\\
&= Ia_i \nabla\cdot\mathbf{B}_0 && (21)\\
&= 0 && (22)
\end{aligned}
$$

since $\nabla\cdot\mathbf{B}=0$ in general.

Thus we get the previous result

$$
\mathbf{F} = \nabla\left(\mathbf{m}\cdot\mathbf{B}\right) \tag{23}
$$