MAGNETOSTATIC FORMULAS FROM ELECTROSTATIC FORMULAS

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.23.

You might have noticed that there are many parallels between the equations of electrostatics and magnetostatics. We can summarize these with the relations between the core quantities in each case.

For electrostatics, in the absence of free charge (although bound charges due to polarization effects may still be present):

$$\nabla\cdot\mathbf{D}=0,\quad\nabla\times\mathbf{E}=0,\quad\varepsilon_0\mathbf{E}=\mathbf{D}-\mathbf{P}\tag{1}$$

For magnetostatics, in the absence of free currents (although again, bound currents due to magnetization may be present):

$$\nabla\cdot\mathbf{B}=0,\quad\nabla\times\mathbf{H}=0,\quad\mu_0\mathbf{H}=\mathbf{B}-\mu_0\mathbf{M}\tag{2}$$

Thus if we have the equations for problems in electrostatics, we should be able to generate the corresponding results for magnetostatics by applying the substitutions:

$$\mathbf{D}\to\mathbf{B},\quad\mathbf{E}\to\mathbf{H},\quad\mathbf{P}\to\mu_0\mathbf{M},\quad\varepsilon_0\to\mu_0$$

Here are 3 examples of how this works.

First, the electric field inside a uniformly polarized sphere is given in Griffiths's example 4.2:

$$\mathbf{E}=-\frac{1}{3\varepsilon_0}\mathbf{P}\tag{3}$$

Making the substitutions, we get the magnetic field inside a uniformly magnetized sphere:

$$\mathbf{H}=-\frac{1}{3}\mathbf{M}\tag{4}$$

$$\mathbf{B}=\mu_0\left(\mathbf{H}+\mathbf{M}\right)=\frac{2}{3}\mu_0\mathbf{M}\tag{5}$$

which agrees with Griffiths's example 6.1.

Second, the electric field inside a sphere of linear dielectric placed in a uniform electric field $\mathbf{E}_0$ is given in Griffiths's example 4.7:

$$\mathbf{E}=\frac{3}{\varepsilon_r+2}\mathbf{E}_0=\frac{3}{3+\chi_e}\mathbf{E}_0 \tag{6}$$

where $\varepsilon_r=1+\chi_e$ is the dielectric constant.

To transform this equation we need to know how to transform χ_e. From its definition, we have

$$\mathbf{P}=\varepsilon_0\chi_e\mathbf{E} \tag{7}$$

Making the substitutions on this equation, we have

$$\mu_0\mathbf{M}=\mu_0 X\mathbf{H} \tag{8}$$

where X is the transformation of χ_e. From the relation between $\mathbf{M}$ and $\mathbf{H}$ in a linear material, we know that $\mathbf{M}=\chi_m\mathbf{H}$, so $X=\chi_m$, and thus the formula for the field inside a sphere of linear magnetic material placed in a uniform field is

$$\mathbf{H}=\frac{3}{3+\chi_m}\mathbf{H}_0 \tag{9}$$

To get this in terms of $\mathbf{B}$, we note that $\mathbf{H}_0=\mathbf{B}_0/\mu_0$ since it is the background field outside the sphere, where $\mathbf{M}=0$. We also have

$$\mathbf{B}=\mu_0\left(\mathbf{H}+\mathbf{M}\right)=\mu_0\left(1+\chi_m\right)\mathbf{H} \tag{10}$$

so

$$\frac{1}{\mu_0\left(1+\chi_m\right)}\mathbf{B} = \frac{3}{3+\chi_m}\frac{1}{\mu_0}\mathbf{B}_0 \tag{11}$$

$$\mathbf{B} = \frac{3+3\chi_m}{3+\chi_m}\mathbf{B}_0 \tag{12}$$

This agrees with the result we found earlier by another method.

Finally, we have the average field within a sphere due to charge distributed arbitrarily within that sphere:

$$\mathbf{E}_{av} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{R^3} \tag{13}$$

where $\mathbf{p}$ is the total dipole moment of the sphere.

Since we know how to transform the polarization density $\mathbf{P}$, and $\mathbf{p} = \frac{4}{3}\pi R^3\mathbf{P}$ where $\mathbf{P}$ is the average polarization density, $\mathbf{p}$ should transform the same way, so

$$\frac{4}{3}\pi R^3\mathbf{P} \rightarrow \frac{4}{3}\pi R^3\mu_0\mathbf{M} = \mu_0\mathbf{m} \tag{14}$$

where $\mathbf{m}$ is the total magnetic dipole moment. Therefore the overall transformation is

$$\mathbf{H}_{av} = -\frac{1}{4\pi R^3}\mathbf{m} \tag{15}$$

$$\mathbf{B}_{av} = \mu_0\left(\mathbf{H}_{av} + \mathbf{M}\right) \tag{16}$$

$$= \mu_0\left(-\frac{1}{4\pi R^3}\mathbf{m} + \frac{3}{4\pi R^3}\mathbf{m}\right) \tag{17}$$

$$= \frac{2\mu_0}{4\pi R^3}\mathbf{m} \tag{18}$$

Again, this agrees with the result we got earlier via another method.