

UNIFORM CHARGE, POLARIZATION & MAGNETIZATION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.24.

If you cast your mind back to the original definition of the electric field, we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' \quad (1)$$

where the integral extends over all space. If ρ is constant, it comes out of the integral:

$$\mathbf{E}(\mathbf{r}) = \frac{\rho}{4\pi\epsilon_0} \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' \equiv \frac{\rho}{4\pi\epsilon_0} \Omega \quad (2)$$

The potential of a polarized object is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' \quad (3)$$

Again, if the polarization is constant, we get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \mathbf{P} \cdot \int \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0} \mathbf{P} \cdot \Omega \quad (4)$$

Finally, for a magnetized object, the vector potential is

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}') d^3\mathbf{r}' \quad (5)$$

For constant magnetization, we get

$$\mathbf{A} = \frac{\mu_0}{4\pi} \mathbf{M} \times \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' = \frac{\mu_0}{4\pi} \mathbf{M} \times \Omega \quad (6)$$

Therefore, if we can work out the electric field for a given shape where the charge density is constant, we can get the potentials of a uniformly polarized or magnetized object of the same shape by simple substitution.

For example, for a charged sphere of radius R , we can find the electric field easily using Gauss's law, and we get

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \frac{\rho}{3\epsilon_0} r \hat{\mathbf{r}} & r < R \\ \frac{\rho R^3}{3\epsilon_0 r^2} \hat{\mathbf{r}} & r > R \end{cases} \quad (7)$$

The potential for a uniformly polarized sphere is found by replacing ρ by \mathbf{P} and taking the dot product, assuming \mathbf{P} points along the z axis:

$$V(\mathbf{r}) = \begin{cases} \frac{r}{3\epsilon_0} \mathbf{P} \cdot \hat{\mathbf{r}} = \frac{r \cos \theta}{3\epsilon_0} P & r < R \\ \frac{R^3}{3\epsilon_0 r^2} \mathbf{P} \cdot \hat{\mathbf{r}} = \frac{R^3 \cos \theta}{3\epsilon_0 r^2} P & r > R \end{cases} \quad (8)$$

The vector potential of a uniformly magnetized sphere is then found by replacing ρ/ϵ_0 by $\mu_0 \mathbf{M}$ and taking the cross product:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 r}{3} \mathbf{M} \times \hat{\mathbf{r}} = \frac{\mu_0 r \sin \theta}{3} M \phi & r < R \\ \frac{\mu_0 R^3}{3r^2} \mathbf{M} \times \hat{\mathbf{r}} = \frac{\mu_0 R^3 \sin \theta}{3r^2} M \phi & r > R \end{cases} \quad (9)$$