UNIFORM CHARGE, POLARIZATION & MAGNETIZATION

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.24.

If you cast your mind back to the original definition of the electric field, we have

$$\mathbf{E}(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int\rho\left(\mathbf{r}'\right)\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}' \tag{1}$$

where the integral extends over all space. If ρ is constant, it comes out of the integral:

$$\mathbf{E}(\mathbf{r})=\frac{\rho}{4\pi\varepsilon_0}\int\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}'\equiv\frac{\rho}{4\pi\varepsilon_0}\Omega \tag{2}$$

The potential of a polarized object is

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int\frac{\left(\mathbf{r}-\mathbf{r}'\right)\cdot\mathbf{P}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}' \tag{3}$$

Again, if the polarization is constant, we get

$$V(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\mathbf{P}\cdot\int\frac{\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}'=\frac{1}{4\pi\varepsilon_0}\mathbf{P}\cdot\Omega \tag{4}$$

Finally, for a magnetized object, the vector potential is

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\mathbf{M}\left(\mathbf{r}'\right)\times\left(\mathbf{r}-\mathbf{r}'\right)d^3\mathbf{r}' \tag{5}$$

For constant magnetization, we get

$$\mathbf{A}=\frac{\mu_0}{4\pi}\mathbf{M}\times\int\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}'=\frac{\mu_0}{4\pi}\mathbf{M}\times\Omega \tag{6}$$

Therefore, if we can work out the electric field for a given shape where the charge density is constant, we can get the potentials of a uniformly polarized or magnetized object of the same shape by simple substitution.

For example, for a charged sphere of radius R, we can find the electric field easily using Gauss's law, and we get

$$\mathbf{E}(\mathbf{r})=\begin{cases}\frac{\rho}{3\varepsilon_0}r\hat{\mathbf{r}} & r<R\\ \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{\mathbf{r}} & r>R\end{cases}\tag{7}$$

The potential for a uniformly polarized sphere is found by replacing ρ by $\mathbf{P}$ and taking the dot product, assuming $\mathbf{P}$ points along the z axis:

$$V(\mathbf{r})=\begin{cases}\frac{r}{3\varepsilon_0}\mathbf{P}\cdot\hat{\mathbf{r}}=\frac{r\cos\theta}{3\varepsilon_0}P & r<R\\ \frac{R^3}{3\varepsilon_0 r^2}\mathbf{P}\cdot\hat{\mathbf{r}}=\frac{R^3\cos\theta}{3\varepsilon_0 r^2}P & r>R\end{cases}\tag{8}$$

The vector potential of a uniformly magnetized sphere is then found by replacing ρ/ε_0 by $\mu_0\mathbf{M}$ and taking the cross product:

$$\mathbf{A}(\mathbf{r})=\begin{cases}\frac{\mu_0 r}{3}\mathbf{M}\times\hat{\mathbf{r}}=\frac{\mu_0 r\sin\theta}{3}M\phi & r<R\\ \frac{\mu_0 R^3}{3r^2}\mathbf{M}\times\hat{\mathbf{r}}=\frac{\mu_0 R^3\sin\theta}{3r^2}M\phi & r>R\end{cases}\tag{9}$$