MAGNETIC DIPOLE EMBEDDED IN SPHERE

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References: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 6.27.

We'll now consider the problem of finding the magnetic field due to an ideal dipole embedded at the centre of a sphere of linear magnetic material with permeability μ.

This is a surprisingly tricky problem, and I must confess that I had to seek a bit of help via everyone's friend: Google. The starting point is to try to figure out the bound currents and calculate the field from them. For a linear material, the magnetization is proportional to the auxiliary field ($\mathbf{M}=\chi_m\mathbf{H}$), so the bound volume current is

$$\mathbf{J}_b=\nabla\times\mathbf{M}=\nabla\times(\chi_m\mathbf{H})=\chi_m\nabla\times\mathbf{H}=\chi_m\mathbf{J}_f \tag{1}$$

where $\mathbf{J}_f$ is the free volume current.

At first glance, it might seem that there is no free current in this problem, so that $\mathbf{J}_b=0$ as well. However, the dipole embedded in the sphere must be caused by a current. (Recall that for a planar loop carrying current I, the dipole moment is $m=IA$, where A is the area enclosed by the loop, and an ideal dipole is just a limiting case as the size of the loop goes to zero, with the current becoming infinite in such a way as to preserve the magnitude of the magnetic moment.) Since the dipole moment is proportional to the current, we can see that the 'free' dipole $\mathbf{m}$ gives rise to a 'bound' dipole $\mathbf{m}_b=\chi_m\mathbf{m}$, also at the centre of the sphere. The effective dipole moment at the centre is therefore the sum of the free and bound dipoles, giving

$$\mathbf{m}_e=\mathbf{m}+\mathbf{m}_b=(1+\chi_m)\,\mathbf{m}=\frac{\mu}{\mu_0}\mathbf{m} \tag{2}$$

Since the embedded dipole is the only free current in the problem, we know that $\mathbf{J}_b=0$ everywhere except at $r=0$, so we can work out the field due to the bound volume current at this point.

We've already worked out the field due to a dipole, so the field due to this effective dipole is

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3}\left[3\left(\mathbf{m}_e\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{m}_e\right] \tag{3}$$

$$= \frac{\mu}{4\pi r^3}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{m}\right] \tag{4}$$

That's the easy bit. The really tricky bit comes when we try to work out the surface bound current $\mathbf{K}_b$. As Griffiths gives the answer in the question, we can see that the field due to the surface current is supposed to be a constant times $\mathbf{m}$. So let's say that the total field is

$$\mathbf{B} = \mathbf{B}_1 + \alpha\mathbf{m} \tag{5}$$

What we need in order to find $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \mathbf{M}\times\hat{\mathbf{r}}$ is the magnetization $\mathbf{M}$. Since the material is linear, we know that $\mathbf{M} = \chi_m\mathbf{H} = \frac{\chi_m}{\mu}\mathbf{B}$ so we get

$$\mathbf{M} = \frac{\chi_m}{4\pi r^3}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{m}\right] + \frac{\alpha\chi_m}{\mu}\mathbf{m} \tag{6}$$

Now at the surface of the sphere, $r = R$ so we get

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{r}} \tag{7}$$

$$= -\frac{\chi_m}{4\pi R^3}\mathbf{m}\times\hat{\mathbf{r}} + \frac{\alpha\chi_m}{\mu}\mathbf{m}\times\hat{\mathbf{r}} \tag{8}$$

$$= \left[\frac{\alpha}{\mu}-\frac{1}{4\pi R^3}\right]\chi_m m\sin\theta\hat{\phi} \tag{9}$$

where the last line follows because we're taking $\mathbf{m} = m\hat{\mathbf{z}}$ and θ is the polar angle in spherical coordinates.

At this point, we have to notice that the form of $\mathbf{K}_b$ is that of a spinning shell with a constant surface charge density σ. This is because in that case, $\mathbf{K}_b = \sigma\mathbf{v}$, where $\mathbf{v}$ is the velocity of a point on the sphere, and for a sphere rotating at constant angular velocity ω this is $\mathbf{v} = \omega\times\mathbf{r} = \omega R\sin\theta\hat{\phi}$. Griffiths shows in his example 5.11 that the field inside the sphere due to this surface current is

$$\mathbf{B} = \frac{2}{3}\mu_0\sigma R\omega\hat{\mathbf{z}} \tag{10}$$

Comparing the two equations, we can make the substitution

$$\sigma\omega R = \left[\frac{\alpha}{\mu}-\frac{1}{4\pi R^3}\right]\chi_m m \tag{11}$$

so the field due to $\mathbf{K}_b$ is

$$\mathbf{B}_2 = \frac{2}{3}\mu_0\left[\frac{\alpha}{\mu}-\frac{1}{4\pi R^3}\right]\chi_m m\hat{\mathbf{z}} \tag{12}$$

$$= \frac{2}{3}\mu_0\left[\frac{\alpha}{\mu}-\frac{1}{4\pi R^3}\right]\chi_m \mathbf{m} \tag{13}$$

In order for this to match up with our original assumption 5, we must have

$$\alpha = \frac{2}{3}\mu_0\left[\frac{\alpha}{\mu}-\frac{1}{4\pi R^3}\right]\chi_m \tag{14}$$

$$\alpha = \frac{2}{3}\frac{\mu_0\chi_m}{4\pi R^3}\left[\frac{2}{3}\frac{\mu_0\chi_m}{\mu}-1\right]^{-1} \tag{15}$$

We can now use $\chi_m = \frac{\mu}{\mu_0}-1$ to get

$$\alpha = \frac{2}{3}\frac{(\mu-\mu_0)}{4\pi R^3}\left[\frac{2}{3}\left(1-\frac{\mu_0}{\mu}\right)-1\right]^{-1} \tag{16}$$

$$= \frac{2}{3}\frac{(\mu-\mu_0)}{4\pi R^3}\left[-\frac{1}{3}-\frac{2\mu_0}{3\mu}\right]^{-1} \tag{17}$$

$$= \frac{2\mu(\mu_0-\mu)}{4\pi R^3(2\mu_0+\mu)} \tag{18}$$

Putting it all together, we get

$$\mathbf{B} = \frac{\mu}{4\pi}\left[\frac{3}{r^3}(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}\right]+\frac{2\mu(\mu_0-\mu)}{4\pi R^3(2\mu_0+\mu)}\mathbf{m} \tag{19}$$

which is the answer given in Griffiths's question.

Outside the sphere, the field is the sum of that due to the effective dipole at the centre and the surface bound charge. Since a spinning spherical shell behaves as an exact dipole when seen from outside, the dipole moment due to the surface charge is, using the substitution 11

$$\mathbf{m}_s = \frac{4}{3}\pi R^3\left[\frac{\alpha}{\mu}-\frac{1}{4\pi R^3}\right]\chi_m\mathbf{m} \tag{20}$$

$$= \frac{1}{3}\left[\frac{2\left(\mu_0-\mu\right)}{2\mu_0+\mu}-1\right]\left(\frac{\mu}{\mu_0}-1\right)\mathbf{m} \tag{21}$$

$$= \frac{\mu\left(\mu_0-\mu\right)}{\mu_0\left(2\mu_0+\mu\right)}\mathbf{m} \tag{22}$$

The total dipole moment seen from outside the sphere is then

$$\mathbf{m}_{out} = \mathbf{m}_s+\mathbf{m}_e \tag{23}$$

$$= \mathbf{m}\frac{\mu}{\mu_0}\left(\frac{\mu_0-\mu}{2\mu_0+\mu}+1\right) \tag{24}$$

$$= \mathbf{m}\frac{3\mu}{2\mu_0+\mu} \tag{25}$$

The field seen outside the sphere is therefore

$$\mathbf{B}_{out} = \frac{\mu_0}{4\pi r^3}\frac{3\mu}{2\mu_0+\mu}\left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}}-\mathbf{m}\right] \tag{26}$$

Now that's what I call a hard problem for this level of textbook. I'd advise Griffiths to mark it with the **!** (more difficult than normal) symbol in the next edition :-)