COAXIAL CABLE WITH VARYING CURRENT

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.16.

Here's another (mis)application of Faraday's law, using the quasistatic approximation to calculate the fields involved. This time we have a coaxial cable in which the current flows in the $+z$ direction on the inner wire and back in the $-z$ direction on the coaxial cylinder. The current varies with time according to

$$I(t) = I_0 \cos\omega t \tag{1}$$

The magnetic field due to the inner wire is given by

$$\mathbf{B}_w(r,t) = \hat{\phi}\frac{\mu_0 I_0}{2\pi r}\cos\omega t \tag{2}$$

The magnetic field due to the coaxial cylinder is just the negative of this outside the cylinder, and zero inside:

$$\mathbf{B}_c(r,t) = \begin{cases} -\hat{\phi}\frac{\mu_0 I_0}{2\pi r}\cos\omega t & r > a \\ 0 & r < a \end{cases} \tag{3}$$

Therefore the net magnetic field outside the cylinder is always zero, and the field inside is given by $\mathbf{B}_w$ alone. The induced electric field is parallel to the z axis, as can be seen by a similar argument to that given for the magnetic field of a solenoid. First, $\mathbf{E}$ cannot have a radial component, since if we reversed the directions of the currents, then $\mathbf{B}$ and hence $\mathbf{E}$ reverse their directions. But reversing the currents is equivalent to turning the wire through 180 degrees, and that shouldn't affect the radial component of the field, so $E_r = 0$.

$\mathbf{E}$ also cannot have a ϕ component. Since $\mathbf{B}$ is entirely in the ϕ direction, so is $-\partial\mathbf{B}/\partial t$, so if we choose a path of integration that is a circle at constant r, and a corresponding surface of integration that is perpendicular to the z direction (that is, to the cable), then $d\mathbf{a}$ is always perpendicular to $-\partial\mathbf{B}/\partial t$ so the integral on the right is zero:

$$\oint \mathbf{E}\cdot d\ell = -\int \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{a} \tag{4}$$

By symmetry, E_ϕ must be constant around the circle so we get $\oint \mathbf{E}\cdot d\ell = 2\pi r E_\phi = 0$, so $E_\phi = 0$. Therefore, $\mathbf{E}$ must have a z component only.

To figure out the actual value of $\mathbf{E}$, we can use the same argument as in calculating the solenoid's magnetic field. We choose a square loop with side length ℓ that lies in the yz plane. First, we take the loop to lie entirely outside the coaxial cable, where $\mathbf{B} = 0$ always. This means that (taking the near edge to be a distance s from the cable's axis):

$$\oint \mathbf{E}\cdot d\ell = \ell\left(E\left(s+\ell\right) - E\left(s\right)\right) = 0 \tag{5}$$

If we assume that $\mathbf{E} \to 0$ at infinity, then $\mathbf{E} = 0$ everywhere outside the cable.

To find $\mathbf{E}$ inside the cable, we can take the loop of integration to have its inner edge at distance s from the axis, where $0 < s < a$, and the outer edge outside the cylinder, where $\mathbf{E} = 0$. Then

$$\begin{aligned}
\oint \mathbf{E}\cdot d\ell &= \ell E\left(s\right) && (6)\\
&= -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} && (7)\\
&= \frac{\ell\omega\mu_0 I_0}{2\pi}\sin\omega t\int_s^a \frac{dr}{r} && (8)\\
&= \frac{\ell\omega\mu_0 I_0}{2\pi}\sin\omega t\ln\frac{a}{s} && (9)
\end{aligned}$$

To get the direction of $\mathbf{E}$ we use Lenz's law and the right hand rule. If the current flows in the $+z$ direction and is increasing, then the magnetic field is in the $+\phi$ direction and increasing, so the flux through the loop is increasing. The induced electric field will oppose this increase, which means that $\mathbf{E}$ inside the cylinder must point in the $-z$ direction. Conversely, if I is decreasing, $\mathbf{E}$ points in the $+z$ direction. Since I starts off at its maximum value at $t = 0$, the current is decreasing initially, so the initial value of $\mathbf{E}$ must point in the $+z$ direction, thus:

$$\mathbf{E}\left(s,t\right) = \begin{cases} \hat{\mathbf{z}}\frac{\omega\mu_0 I_0}{2\pi}\sin\omega t\ln\frac{a}{s} & 0 < s < a \\ 0 & s > a \end{cases} \tag{10}$$

As always, we need to issue a caution that this analysis is deeply flawed, since we're applying the quasistatic approximation to a cable of infinite length, and we're assuming that the current can change everywhere along the length of the cable at the same rate, which doesn't happen in real life.

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