MUTUAL INDUCTANCE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.20.

One consequence of Faraday's law is that if we have two separate circuits in close proximity to each other, a varying current in one circuit will induce an emf in the other circuit, and vice versa. This is the principle used in the electrical transformers commonly found on power cords for household electronic devices. This phenomenon is known as *induction*. We'll begin by looking at the magnetic flux through one circuit by a steady current in the other.

The Biot-Savart law says that the magnetic field produced by a steady linear current I_1 is

$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0 I_1}{4\pi}\int\frac{d\ell_1\times(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3} \tag{1}$$

where the integral is taken around circuit number 1. For any given circuit, this integral will probably be very difficult to work out, but the important thing is that, for a steady current, $\mathbf{B}_1$ is proportional to that current. This means that the flux through the other circuit due to this field is also proportional to I_1, since the flux through circuit number 2 due to $\mathbf{B}_1$ is

$$\Phi_{21} = \int\mathbf{B}_1\cdot d\mathbf{a}_2 \tag{2}$$

where the integral is over the area enclosed by circuit 2.

From the Biot-Savart law and the flux equation we can see that apart from the proportionality to I_1, everything else (well, apart from the constant μ_0) that goes into the formula for flux depends only on the geometry of the two loops. This 'everything else' is called the *mutual inductance* M_{21}, that is

$$\Phi_{21} = M_{21}I_1 \tag{3}$$

with

$$M_{21}=\frac{\mu_0}{4\pi}\int\left[\int\frac{d\ell_1\times(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}\right]\cdot d\mathbf{a}_2 \tag{4}$$

The curious thing is that the mutual inductance of circuit 1 relative to circuit 2 is the same, that is

$$M_{12}=M_{21} \tag{5}$$

This isn't exactly obvious from the double integral above, but we can see it as follows. We can write 2 in terms of the magnetic vector potential as follows:

$$\Phi_{21}=\int(\nabla\times\mathbf{A}_1)\cdot d\mathbf{a}_2 \tag{6}$$

Using Stokes's theorem, this becomes a line integral:

$$\int(\nabla\times\mathbf{A}_1)\cdot d\mathbf{a}_2=\int\mathbf{A}_1\cdot d\ell_2 \tag{7}$$

From our discussion of the vector potential, we have the formula:

$$\mathbf{A}_1(\mathbf{r}_2)=\frac{\mu_0 I_1}{4\pi}\int\frac{d\ell_1}{|\mathbf{r}_2-\mathbf{r}_1|} \tag{8}$$

so

$$\Phi_{21}=\frac{\mu_0 I_1}{4\pi}\int\int\frac{d\ell_1\cdot d\ell_2}{|\mathbf{r}_2-\mathbf{r}_1|} \tag{9}$$

From this formula, it's obvious that if the same current I_1 flows in either circuit, the flux through the other circuit is the same, since the integrand is identical under interchange of the subscripts 1 and 2.

As one example that this is true, suppose we have a circular loop of radius b in the xy plane with its centre at the origin, and a second loop of radius $a\ll b$ parallel to the first with its centre on the z axis a distance z above the first loop. If a current I flows through the larger loop, what is the flux through the smaller one?

The magnetic field at a point on the z axis above a current loop is given by Griffiths in his example 5.6 (for a similar derivation involving the electric field produced by a toroidal solenoid, see here), and is

(10)
$$B_1 = \frac{\mu_0 I}{2} \frac{b^2}{(b^2+z^2)^{3/2}}$$

Since $a \ll b$, we can take B_1 to be approximately constant over the smaller loop, and so the flux through it is

(11)
$$\Phi_{21} = B_1 \pi a^2 = \frac{\mu_0 I}{2} \frac{\pi a^2 b^2}{(b^2+z^2)^{3/2}}$$

Now suppose the same current flows through the smaller loop. What is the flux through the larger one? This time we can't take the field at the larger loop to be constant, but since the smaller loop is *really* small, we can approximate it by a magnetic dipole, of dipole moment $\mathbf{m} = I\pi a^2 \hat{\mathbf{z}}$. The formula for the field due to a dipole is

(12)
$$\mathbf{B}_2 = \frac{\mu_0}{4\pi r^3} \left[3\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{m}\right]$$

To get the flux, we need to integrate this over the area enclosed by the larger loop. The distance r is the distance from the upper dipole to a point in the area enclosed by the lower loop. If this point is at radial distance s from the centre of the loop, and the two loops are a distance z apart, then

(13)
$$r = \sqrt{s^2+z^2}$$

We can also define θ as the angle between $\hat{\mathbf{z}}$ and the radial vector $\hat{\mathbf{r}}$. Then

(14)
$$\mathbf{m}\cdot\hat{\mathbf{r}} = I\pi a^2 \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = I\pi a^2 \cos\theta = I\pi a^2 \frac{z}{\sqrt{s^2+z^2}}$$

Therefore the field is

(15)
$$\mathbf{B}_2 = \frac{\mu_0 \pi a^2 I}{4\pi\left(s^2+z^2\right)^{3/2}} \left[\frac{3z}{\sqrt{s^2+z^2}}\hat{\mathbf{r}} - \hat{\mathbf{z}}\right]$$

Now if we integrate this over the area of circuit b, only the component of $\hat{\mathbf{r}}$ in the z direction will contribute, as the other components cancel out due to symmetry. Thus we'll get another factor of $\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}$ so

$$\Phi_{12} = \int \mathbf{B}_2 \cdot d\mathbf{a} = \frac{\mu_0 a^2 I}{4}\left[3z^2 \int_0^b \frac{2\pi s ds}{\left(s^2+z^2\right)^{5/2}} - \int_0^b \frac{2\pi s ds}{\left(s^2+z^2\right)^{3/2}}\right] \tag{16}$$

$$= \frac{\mu_0 \pi a^2 I}{2}\left[\left.\frac{-z^2}{\left(s^2+z^2\right)^{3/2}}\right|_0^b + \left.\frac{1}{\sqrt{s^2+z^2}}\right|_0^b\right] \tag{17}$$

$$= \frac{\mu_0 \pi a^2 I}{2}\left[\left.\frac{-z^2+s^2+z^2}{\left(s^2+z^2\right)^{3/2}}\right|_0^b\right] \tag{18}$$

$$= \frac{\mu_0 I}{2}\frac{\pi a^2 b^2}{\left(b^2+z^2\right)^{3/2}} = \Phi_{21} \tag{19}$$

Thus the two fluxes are indeed the same. The mutual inductance is therefore the same both ways, and we can drop the subscripts and just refer to it as M:

$$M = \frac{\mu_0}{2}\frac{\pi a^2 b^2}{\left(b^2+z^2\right)^{3/2}} \tag{20}$$

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