LC CIRCUIT - THE OSCILLATOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.25.

Another inductance problem is that of the oscillator circuit consisting of a capacitor and an inductor, with a switch in the circuit that can be opened or closed. The capacitor is charged initially to a potential V_0 and then the switch is closed. What happens?

It's important to be clear about the directions (and thus the signs) of the charge and current in this problem. To be definite, suppose we draw the circuit as a square with the inductor on the left edge and the capacitor on the right. The capacitor starts off with a charge $+Q$ on its lower plate, and we define the positive direction of current to be clockwise. When the switch is closed, the current starts to flow from the positive to the negative plate of the capacitor, so the initial current is clockwise (positive) and increasing. However, the charge on the capacitor is decreasing, so $\dot{Q}<0$. Since the rate of change of charge is (in magnitude) equal to the current, we must have

$$I=-\dot{Q} \tag{1}$$

The back emf provided by the inductor must oppose this current, so since $\dot{I}>0$ initially, we have

$$\mathcal{E}_L=-L\dot{I}=+L\ddot{Q} \tag{2}$$

and the total emf in the circuit is the sum of the contributions from the inductor and capacitor, or

$$\mathcal{E}=\mathcal{E}_C+\mathcal{E}_L=\frac{Q}{C}+L\ddot{Q} \tag{3}$$

Since there is no resistor in the circuit, the total emf around the circuit must be zero, so we get the differential equation for Q:

$$\frac{Q}{C}+L\ddot{Q}=0 \tag{4}$$

Note that we should always get the same ODE no matter what directions we specify for the current. For example, if we had defined positive current as in the counterclockwise direction, and started off with the capacitor in the same configuration as before, then initially we'd have $I=+\dot{Q}$. However, this time $\dot{I}(0)<0$ since the current is becoming more negative (it's still increasing in magnitude but in the negative direction), so this time $\mathcal{E}_L=+L\dot{I}$ and we get the same ODE as before.

The ODE can be written in the more familiar form

$$\ddot{Q}=-\frac{1}{LC}Q\equiv-\omega^2Q \tag{5}$$

where $\omega=1/\sqrt{LC}$. This has the general solution

$$\begin{aligned} Q(t) &= A\sin\omega t+B\cos\omega t & (6)\\ I(t)=-\dot{Q} &= -\omega A\cos\omega t+\omega B\sin\omega t & (7)\end{aligned}$$

with initial conditions

$$\begin{aligned} Q(0) &= CV_0 & (8)\\ \dot{Q}(0)=-I(0) &= 0 & (9)\end{aligned}$$

from which we get $A=0$ and $B=CV_0$. Thus the current is

$$I(t)=V_0\sqrt{\frac{C}{L}}\sin\frac{t}{\sqrt{LC}} \tag{10}$$

Thus the current oscillates with angular frequency $1/\sqrt{LC}$.

If we add a resistor R in series with L and C, then the combined emf of L and C produces a voltage of $IR=-\dot{Q}R$ and the ODE now becomes

$$\begin{aligned} \frac{Q}{C}+L\ddot{Q} &= -\dot{Q}R & (11)\\ L\ddot{Q}+R\dot{Q}+\frac{Q}{C} &= 0 & (12)\end{aligned}$$

We can get the solution to this ODE by proposing $Q=Ae^{\alpha t}$ and substituting. We get, after cancelling off the factors of $Ae^{\alpha t}$, the quadratic equation

$$
\begin{aligned}
L\alpha^2 + R\alpha + \frac{1}{C} &= 0 && (13)\\
\alpha &= \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} && (14)\\
&= -\frac{R}{2L}\left(1 \pm \sqrt{1 - \frac{4L}{R^2 C}}\right) && (15)
\end{aligned}
$$

If $R > 2\sqrt{L/C}$ then α is real and both roots are negative, so the current decays exponentially to zero. If $R < 2\sqrt{L/C}$, the roots are complex, so there will be an exponential decay factor and an oscillatory factor, so the current is a damped oscillation that decays to zero.