ENERGY IN A MAGNETIC FIELD

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.26.

We've seen that the energy stored in an electric field is

$$W_E = \frac{\varepsilon_0}{2}\int E^2 d^3\mathbf{r} \tag{1}$$

where the integral is over all space. Here we'll look at the derivation of a similar formula for the magnetic field.

The magnetic flux through an inductor carrying current I is

$$\Phi = LI \tag{2}$$

where L is the inductance of the circuit. The emf induced in a circuit by changing the current in that circuit is then

$$\mathscr{E} = -\frac{d\Phi}{dt} = -L\dot{I} \tag{3}$$

The power generated by a current is the voltage multiplied by the current, and this power is the rate at which work is done, so

$$\frac{dW_B}{dt} = -\mathscr{E}I = LI\dot{I} \tag{4}$$

The sign in this equation reflects the fact that increasing the current through the inductor induces an emf $\mathscr{E}$ that opposes the increase, so we need to do work against this back emf to generate the current. This work should be positive for an increasing current (that is, for $\dot{I} > 0$), so we've defined the signs to make this true.

If we increase the current from zero to some final value I then by integrating this equation we get the total work done:

$$W_B = \frac{1}{2}LI^2 \tag{5}$$

This formula shows that the total energy delivered to the inductor depends only on the final current, and not the route by which we get there. Thus the current could increase for a while, then decrease for a bit, and then increase again, but as long as it ends up at the final value of I the same work is required.

To convert this expression to one containing the magnetic field requires a bit of juggling with vector calculus, so here we go. First, we can write LI in terms of $\mathbf{B}$:

$$LI = \Phi = \int \mathbf{B} \cdot d\mathbf{a} \tag{6}$$

In terms of the vector potential, this becomes, using Stokes's theorem:

$$LI = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} \tag{7}$$

$$= \oint \mathbf{A} \cdot d\ell \tag{8}$$

where we've converted to a line integral around the circuit bordering the area. Therefore, the work is

$$W_B = \frac{1}{2} I \oint \mathbf{A} \cdot d\ell = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I})\, d\ell \tag{9}$$

where we've made the current a vector in place of the directed line element. If the current occupies a volume rather than a linear circuit, we can write the generalization of this as

$$W_B = \frac{1}{2} \int (\mathbf{A} \cdot \mathbf{J})\, d^3\mathbf{r} \tag{10}$$

If these are steady currents (that is, we've increased the current up to a certain value and then held it there), we can apply Ampère's law in the form $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ to get

$$W_B = \frac{1}{2\mu_0} \int \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d^3\mathbf{r} \tag{11}$$

Now we can use a vector calculus identity:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{12}$$

to write

$$W_B = \frac{1}{2\mu_0}\int \mathbf{B}\cdot(\nabla\times\mathbf{A})\,d^3\mathbf{r} - \frac{1}{2\mu_0}\int \nabla\cdot(\mathbf{A}\times\mathbf{B})\,d^3\mathbf{r} \tag{13}$$

$$= \frac{1}{2\mu_0}\int B^2 d^3\mathbf{r} - \frac{1}{2\mu_0}\int \nabla\cdot(\mathbf{A}\times\mathbf{B})\,d^3\mathbf{r} \tag{14}$$

We can use the divergence theorem to convert the second integral to a surface integral, and then perform the usual trick of letting the surface go to infinity. We get

$$W_B = \frac{1}{2\mu_0}\int B^2 d^3\mathbf{r} - \frac{1}{2\mu_0}\int (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a} \tag{15}$$

If the currents are all localized, then both **A** and **B** tend to zero at infinity, so we can ignore this final integral and get

$$W_B = \frac{1}{2\mu_0}\int B^2 d^3\mathbf{r} \tag{16}$$

This is the energy stored in a (localized) magnetic field produced by steady currents.

As an example, we'll consider the standard case of the infinite solenoid (I know, I know, we derived this formula for *finite* current distributions, so you can think of this problem as a 'very long' solenoid rather than an infinite one), with n turns per unit length carrying current I. We'll work out the energy per unit length of a section far from the ends.

If the solenoid has radius R, its inductance per unit length is

$$L = \pi R^2 \mu_0 n^2 \tag{17}$$

so from 5 we have

$$W_B = \frac{1}{2}\pi R^2 \mu_0 n^2 I^2 \tag{18}$$

We can also work this out from 9 using the vector potential of the solenoid given by Griffiths in his example 5.12:

$$\mathbf{A} = \frac{\mu_0 n I r}{2}\hat{\phi} \tag{19}$$

for $r<R$.

In a solenoid, $\mathbf{I}$ is in the ϕ direction so the total current in unit length is $nI\hat{\phi}$, the line integral goes around the solenoid at $r=R$ and we get

$$\frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})\,d\ell=\frac{1}{2}\frac{\mu_0 nIR}{2}2\pi RnI=\frac{1}{2}\pi R^2\mu_0 n^2 I^2 \tag{20}$$

Next, we can use 16. The field inside the solenoid is a constant

$$B=n\mu_0 I \tag{21}$$

and the volume per unit length is πR^2 so

$$W_B=\frac{1}{2\mu_0}\pi R^2\left(n\mu_0 I\right)^2=\frac{1}{2}\pi R^2\mu_0 n^2 I^2 \tag{22}$$

Finally, we can use 15. Instead of taking the integration volume to be all space, we can use any volume that completely encloses the current, so we can use a cylindrical tube of inner radius $r=a<R$ to $r=b>R$. Outside the solenoid, $\mathbf{B}=0$ so we need look only at the region $a\le r\le R$. The first integral is

$$\frac{1}{2\mu_0}\int_{a\le r\le R}B^2 d^3\mathbf{r}=\frac{1}{2}\pi\left(R^2-a^2\right)\mu_0 n^2 I^2 \tag{23}$$

To do the surface integral, we first work out the direction of $\mathbf{A}\times\mathbf{B}$. $\mathbf{A}$ is in the ϕ direction and $\mathbf{B}$ is in the z direction, so $\mathbf{A}\times\mathbf{B}$ is in the radial direction, pointing outwards. On the inner surface of the tube, $d\mathbf{a}$ points radially inwards, so $(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}<0$, and the integral is

$$-\frac{1}{2\mu_0}\int_{r=a}(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}=\frac{1}{2\mu_0}2\pi a\frac{\mu_0 nIa}{2}n\mu_0 I=\frac{1}{2}\pi a^2\mu_0 n^2 I^2 \tag{24}$$

Adding the two contributions, we get

$$W_B=\frac{1}{2}\pi\left(R^2-a^2\right)\mu_0 n^2 I^2+\frac{1}{2}\pi a^2\mu_0 n^2 I^2=\frac{1}{2}\pi R^2\mu_0 n^2 I^2 \tag{25}$$

Thus all four methods give the same answer.

PINGBACKS

Pingback: Energy in a toroidal solenoid
Pingback: Inductance of a coaxial cable
Pingback: LR circuit
Pingback: Poynting's theorem
Pingback: Energy flow in time-dependent fields
Pingback: Magnetic systems in thermodynamics