MUTUAL INDUCTANCE BETWEEN TWO MAGNETIC DIPOLES

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.30.

This problem relates mutual inductance to the interaction energy of two magnetic dipoles. Suppose we have two tiny loops of wire with area vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. The loops become dipoles if they carry a current I so that their magnetic dipole moments are $\mathbf{m}_i = I\mathbf{a}_i$ $(i = 1,2)$. We can work out the mutual inductance between these two loops if we assume that the areas are small enough that the field produced by one dipole is constant over the area of the other dipole. The magnetic field of dipole 2 as felt by dipole 1 is

$$\mathbf{B}_{21} = \frac{\mu_0}{4\pi r_{12}^3}\left[3\left(\mathbf{m}_2\cdot\hat{\mathbf{r}}_{21}\right)\hat{\mathbf{r}}_{21}-\mathbf{m}_2\right] \tag{0.1}$$

$$= \frac{\mu_0 I}{4\pi r_{12}^3}\left[3\left(\mathbf{a}_2\cdot\hat{\mathbf{r}}_{21}\right)\hat{\mathbf{r}}_{21}-\mathbf{a}_2\right] \tag{0.2}$$

where $\hat{\mathbf{r}}_{21}$ is the unit vector pointing from dipole 2 to dipole 1. The mutual inductance can be worked out if we know the flux through dipole 1. With our approximation of constant field over dipole 1, we get

$$\Phi_{21} = \mathbf{B}_{21}\cdot\mathbf{a}_1 \tag{0.3}$$

$$= \frac{\mu_0 I}{4\pi r_{12}^3}\left[3\left(\mathbf{a}_2\cdot\hat{\mathbf{r}}_{21}\right)\left(\hat{\mathbf{r}}_{21}\cdot\mathbf{a}_1\right)-\mathbf{a}_2\cdot\mathbf{a}_1\right] \tag{0.4}$$

The mutual inductance is

$$M_{21} = \frac{\Phi_{21}}{I} = \frac{\mu_0}{4\pi r_{12}^3}\left[3\left(\mathbf{a}_2\cdot\hat{\mathbf{r}}_{21}\right)\left(\hat{\mathbf{r}}_{21}\cdot\mathbf{a}_1\right)-\mathbf{a}_2\cdot\mathbf{a}_1\right] \tag{0.5}$$

Conversely, if we run a current I through dipole 2 and look at the flux through dipole 1, we get

$$\Phi_{12} = \mathbf{B}_{12}\cdot\mathbf{a}_2 \tag{0.6}$$

$$= \frac{\mu_0 I}{4\pi r_{12}^3}\left[3\left(\mathbf{a}_1\cdot\hat{\mathbf{r}}_{12}\right)\left(\hat{\mathbf{r}}_{12}\cdot\mathbf{a}_2\right)-\mathbf{a}_1\cdot\mathbf{a}_2\right] \tag{0.7}$$

Since $\hat{\mathbf{r}}_{12} = -\hat{\mathbf{r}}_{21}$, we see that $\Phi_{12} = \Phi_{21}$ and thus $M_{12} = M_{21}$ as required.

If we now start off with a current I_1 flowing in dipole 1 but no current in dipole 2, then switch on a current I_2 in dipole 2, the emf induced in dipole 1 is given by the change in flux:

$$\mathscr{E}_{21} = -\frac{d\Phi_{21}}{dt} \tag{0.8}$$

$$= -M_{21}\frac{dI_2}{dt} \tag{0.9}$$

$$= -\frac{\mu_0}{4\pi r_{12}^3}\frac{dI_2}{dt}\left[3\left(\mathbf{a}_2\cdot\hat{\mathbf{r}}_{21}\right)\left(\hat{\mathbf{r}}_{21}\cdot\mathbf{a}_1\right) - \mathbf{a}_2\cdot\mathbf{a}_1\right] \tag{0.10}$$

To get the total work required if we want to maintain I_1 in dipole 1, we need to consider how this induced emf affects I_1. If we didn't attempt to maintain I_1, then I_1 would change in the direction prescribed by the sign of $\mathscr{E}_{21}$. So if we want to maintain I_1, we have to oppose $\mathscr{E}_{21}$, and thus the rate at which the work is done is $dW_{21}/dt = -\mathscr{E}_{21}I_1$, where the minus sign indicates we're opposing $\mathscr{E}_{21}$.

$$\frac{dW_{21}}{dt} = -\mathscr{E}_{21}I_1 \tag{0.11}$$

$$= \frac{\mu_0 I_1}{4\pi r_{12}^3}\frac{dI_2}{dt}\left[3\left(\mathbf{a}_2\cdot\hat{\mathbf{r}}_{21}\right)\left(\hat{\mathbf{r}}_{21}\cdot\mathbf{a}_1\right) - \mathbf{a}_2\cdot\mathbf{a}_1\right] \tag{0.12}$$

Since we're keeping I_1 and all the geometrical terms constant, we can just integrate this expression to get the total work done:

$$W_{21} = \frac{\mu_0 I_1}{4\pi r_{12}^3}\left[3\left(\mathbf{a}_2\cdot\hat{\mathbf{r}}_{21}\right)\left(\hat{\mathbf{r}}_{21}\cdot\mathbf{a}_1\right) - \mathbf{a}_2\cdot\mathbf{a}_1\right]\int_0^t \frac{dI_2}{dt}dt \tag{0.13}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi r_{12}^3}\left[3\left(\mathbf{a}_2\cdot\hat{\mathbf{r}}_{21}\right)\left(\hat{\mathbf{r}}_{21}\cdot\mathbf{a}_1\right) - \mathbf{a}_2\cdot\mathbf{a}_1\right] \tag{0.14}$$

$$= \frac{\mu_0}{4\pi r_{12}^3}\left[3\left(\mathbf{m}_2\cdot\hat{\mathbf{r}}_{21}\right)\left(\hat{\mathbf{r}}_{21}\cdot\mathbf{m}_1\right) - \mathbf{m}_2\cdot\mathbf{m}_1\right] \tag{0.15}$$

This last expression is exactly equal to the interaction energy between the two dipoles except that it's opposite in sign. I'm not really sure why this should be (assuming this answer is correct), though the two cases are physically different. In this case, we start with one dipole at full strength given by $\mathbf{m}_1$ and then turn the other one on, so the work doesn't involve any movement of the dipoles; we can move dipole 2 into position before we turn the current on, so there is no interaction with the first dipole until it's in place. In the previous case, both dipoles were switched on and then moved into position, so the motion of the dipoles causes the change in flux (and thus the back emf), rather than any change in the current (and hence in the dipoles' strengths). Not sure.