DISPLACEMENT CURRENT IN A CAPACITOR

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.32.

Here's a slight variant on the previous problem in which we considered a thick wire of radius a with a small gap of width $w \ll a$. A steady current I flows along the wire with the result that the gap acts like a parallel plate capacitor.

This time, we'll replace the thick wire with a very thin wire and replace the gap with two circular plates, again with radius a and separated by a gap of width $w \ll a$, so we have a more realistic capacitor. We still have a steady current I, and we'll assume that charge piles up on each plate in such a way that the surface charge density σ is constant in space (obviously σ is increasing with time).

The electric field between the plates is the same as in the previous problem, so

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{z}} \tag{0.1}$$

Now σ is increasing as current flows onto the plates, so

$$\begin{aligned} \frac{\partial E}{\partial t} &= \frac{1}{\epsilon_0}\frac{\partial \sigma}{\partial t} && (0.2)\\ &= \frac{1}{\pi a^2 \epsilon_0}\frac{dQ}{dt} && (0.3)\\ &= \frac{I}{\pi a^2 \epsilon_0} && (0.4) \end{aligned}$$

and the field as a function of time is

$$\mathbf{E} = \frac{It}{\pi a^2 \epsilon_0}\hat{\mathbf{z}} \tag{0.5}$$

The displacement current is then

$$\mathbf{J}_d = \varepsilon_0\dot{\mathbf{E}} \tag{0.6}$$
$$= \frac{\varepsilon_0}{\pi a^2\varepsilon_0}\frac{dQ}{dt}\hat{\mathbf{z}} \tag{0.7}$$
$$= \frac{I}{\pi a^2}\hat{\mathbf{z}} \tag{0.8}$$

To find the induced magnetic field between the plates we can take as the path of integration a circle of radius r halfway between the plates, and the calculation is the same as in the previous post if we take the area of integration to be the flat circular area filling the circle. Thus we get

$$\mathbf{B} = \mu_0 I\frac{r}{2\pi a^2}\hat{\phi} \tag{0.9}$$

However, in applying Stokes's law, we are allowed to choose as the area bounded by the curve any area we like, not necessarily a planar one. So we can choose an open-ended cylinder with the open end bounded by the circle between the plates, the sides of the cylinder extending back through one of the plates to a point behind the plate, and closed off with a circular cap that cuts through the wire leading up to the plate. Since only part of the sides of the cylinder project into the gap between the plates, and the normal to this area is perpendicular to $\mathbf{J}_d$, there is no contribution to the area integral from the displacement current.

Behind the plate, we can apply the original form of Ampère's law since the currents here are all steady. There is a current I crossing into the surface where the wire cuts the end cap, but if $r < a$, current is also flowing *out* of the cylinder as it spreads out over the plate. Thus the net current enclosed by the surface is

$$I_{enc} = I - I\frac{\pi\left(a^2 - r^2\right)}{\pi a^2} = I\frac{r^2}{a^2} \tag{0.10}$$

The line integral of the magnetic field is the same as before (since the circle around which the integral is taken hasn't changed), so

$$2\pi rB = \mu_0 I_{enc} \tag{0.11}$$
$$\mathbf{B} = \mu_0 I\frac{r}{2\pi a^2}\hat{\phi} \tag{0.12}$$

so we get the same answer whatever surface we choose.