MAXWELL'S EQUATIONS AND AN EXPANDING SHELL OF CHARGE

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Problem 7.34.

We can now summarize Maxwell's equations

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{2}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{3}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{4}$$

A novel example of electric and magnetic fields that satisfy these equations is

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}H(vt-r)\hat{\mathbf{r}} \tag{5}$$

$$\mathbf{B} = 0 \tag{6}$$

where $H(x)$ is the step function

$$H(x)\equiv\begin{cases}1 & x>0\\ 0 & x<0\end{cases} \tag{7}$$

(I've used H instead of Griffiths's θ for the step function to avoid confusion with the spherical angle θ.)

Fairly obviously, $\nabla\cdot\mathbf{B}=\nabla\times\mathbf{B}=0$ and, since $\mathbf{E}$ has only a radial component which depends only on r, $\nabla\times\mathbf{E}=0$ (if you look up the form of the curl in spherical coordinates, the only derivatives of E_r are with respect to θ and ϕ which are both zero).

You might think that $\nabla\cdot\mathbf{E}$ is fairly easy to calculate using the definition of the divergence in spherical coordinates, but we must remember that something bizarre happens at $r=0$; we must use the formula for the 3-d delta function:

$$\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^{2}}\right)=4\pi\delta^{3}\left(\mathbf{r}\right) \tag{8}$$

and the product rule for the divergence of the product of a scalar field f and a vector field $\mathbf{A}$:

$$\nabla\cdot\left(f\mathbf{A}\right)=\mathbf{A}\cdot\nabla f+f\nabla\cdot\mathbf{A} \tag{9}$$

Here we have

$$\begin{aligned} f &= H\left(vt-r\right) && (10)\\ \mathbf{A} &= -\frac{1}{4\pi\varepsilon_{0}}\frac{q}{r^{2}}\hat{\mathbf{r}} && (11)\end{aligned}$$

Using the derivative of the step function $dH\left(x\right)/dx=\delta\left(x\right)$ we have

$$\frac{dH\left(vt-r\right)}{dr}=-\delta\left(vt-r\right) \tag{12}$$

so

$$\begin{aligned}\nabla\cdot\mathbf{E} &= -\frac{1}{4\pi\varepsilon_{0}}\frac{q}{r^{2}}\hat{\mathbf{r}}\cdot\left(\nabla H\left(vt-r\right)\right)-H\left(vt-r\right)\frac{q}{4\pi\varepsilon_{0}}\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^{2}}\right) && (13)\\ &= \frac{1}{4\pi\varepsilon_{0}}\frac{q}{r^{2}}\hat{\mathbf{r}}\cdot\left(\delta\left(vt-r\right)\hat{\mathbf{r}}\right)-H\left(vt-r\right)\frac{q}{\varepsilon_{0}}\delta^{3}\left(\mathbf{r}\right) && (14)\end{aligned}$$

The second term is non-zero only at $\mathbf{r}=0$ so, assuming $vt>0$ we can omit the step function since it is always 1 at $\mathbf{r}=0$. We therefore have

$$\nabla\cdot\mathbf{E}=\frac{q}{4\pi r^{2}\varepsilon_{0}}\delta\left(vt-r\right)-\frac{q}{\varepsilon_{0}}\delta^{3}\left(\mathbf{r}\right) \tag{15}$$

The charge density is therefore

$$\begin{aligned}\rho &= \varepsilon_{0}\nabla\cdot\mathbf{E} && (16)\\ &= \frac{q}{4\pi r^{2}}\delta\left(vt-r\right)-q\delta^{3}\left(\mathbf{r}\right) && (17)\end{aligned}$$

This corresponds to a point charge $-q$ at the origin and an expanding (with speed v) spherical shell of charge of total amount $+q$. The electric field inside the shell is due entirely to the point charge at the origin, and the field outside the shell is zero since the total enclosed charge is zero.

The current density is obtained from $\nabla\times\mathbf{B}=0$, so:

$$\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} = 0 \tag{18}$$

$$\mathbf{J} = -\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{19}$$

$$= \frac{qv}{4\pi r^2}\delta\left(vt-r\right)\hat{\mathbf{r}} \tag{20}$$

The expanding shell provides the current density; the stationary charge at the origin doesn't contribute.

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